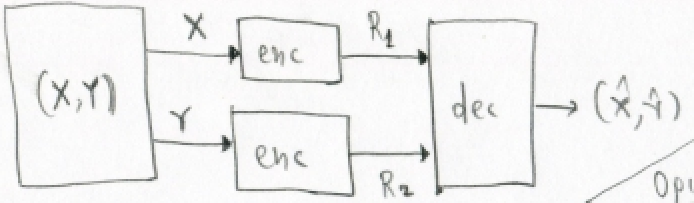
Seminar 3

Info Theory July 2009

Encoding of correlated sources

– Αν έχουμε μία πηγή, τότε $R > H(X)$ είναι καλό για αυθαίρετα ακριβή σημείου.



– separate encoding of correlated sources.

– obvious encoding: $R > H(X) + H(Y)$.

– Slepian, Wolf: $R > H(X,Y)$ sufficient.

Έστω $(X_1,Y_1),\dots,(X_n,Y_n)$ iid $\sim p(x,y)$.

<u>Ορισμός:</u> $((2^{nR_1}, 2^{nR_2}), n)$ distributed source code for joint source (X,Y) consists of

$$f_1 : \mathcal{X}^n \to \{1,\dots,2^{nR_1}\}$$

$$f_2 : \mathcal{Y}^n \to \{1,\dots,2^{nR_2}\}$$

$$g : \{1,\dots,2^{nR_1}\} \times \{1,\dots,2^{nR_2}\} \to \mathcal{X}^n \times \mathcal{Y}^n$$

<u>Ορισμός:</u> Πιθανότητα σφάλματος

$$P_e^{(n)} = P\left[g(f_1(X_1^n), f_2(X_2^n)) \neq (X^n, Y^n)\right]$$

<u>Ορισμός:</u> Το ζεύγος (R_1, R_2) καλείται επιτεύξιμο αν $\exists$ ακολουθία $((2^{nR_1}, 2^{nR_2}), n)$-κωδίκων με $P_e^{(n)} \to 0$ όταν $n \to \infty$. Η περιοχή επιτεύξιμων ρυθμών είναι η θήκη του συνόλου των επιτεύξιμων ρυθμών.

Θεώρημα Slepian-Wolf: (X,Y) iid $\sim p(x,y)$

$$R_1 > H(X|Y), \quad R_2 > H(Y|X), \quad R_1 + R_2 > H(X,Y)$$

Random binning: partition $\mathcal{X}^n$ into 2^{nR_1} bins and $\mathcal{Y}^n$ into 2^{nR_2} bins.

Random code generation: Assign every $x^n \in \mathcal{X}^n$ to one of 2^{nR_1} bins independently and uniformly. Similarly assign every $y^n \in \mathcal{Y}^n$ to one of 2^{nR_2} bins. Reveal the assignments f_1 and f_2 to the encoder and the decoder

<u>Encoding</u>: of (X^n, Y^n).

sender 1 sends the bin to which X^n belongs

" 2 Y^n " .

<u>Decoding</u>: Given the index pair (i_0, j_0), declare

$(\hat{x}^n, \hat{y}^n) = (x^n, y^n)$ if there is 1 and only 1 pair of sequences (x^n, y^n) such that $f_1(x^n) = i_0$ and $f_2(y^n) = j_0$ and $(x^n, y^n) \in A_\epsilon^{(n)}$. Otherwise, declare error.

<u>Probability of error</u>: $(X_i, Y_i) \sim p(x,y)$.

$$E_0 = \{(X^n, Y^n) \notin A_\epsilon^{(n)}\}$$

$$E_1 = \{\exists\, x^{n'} \neq X^n : f_1(x^{n'}) = f_1(X^n) \text{ and } (x^{n'}, Y^n) \in A_\epsilon^{(n)}\}$$

$$E_2 = \{\exists\, y^{n'} \neq Y^n : f_2(y^{n'}) = f_2(Y^n) \text{ and } (X^n, y^{n'}) \in A_\epsilon^{(n)}\}$$

$$E_{12} = \{\exists\, (x^{n'}, y^{n'}) : x^{n'} \neq X^n,\ y^{n'} \neq Y^n,\ f_1(x^{n'}) = f_1(X^n),\ f_2(y^{n'}) = f_2(Y^n) \text{ and } (x^{n'}, y^{n'}) \in A_\epsilon^{(n)}\}$$

X^n, Y^n, f_1 and f_2 are random

$$P_e^{(n)} = P(E_0 \cup E_1 \cup E_2 \cup E_{12}) \leq P(E_0) + P(E_1) + P(E_2) + P(E_{12})$$

$- P(E_0) \leq \epsilon$

$$- P(E_1) = \sum_{(x^n, y^n)} p(x^n, y^n)\, P\left[\exists\, x^{n'} \neq x^n : f_1(x^{n'}) = f_1(x^n),\ (x^{n'}, y^n) \in A_\epsilon^{(n)}\right]$$

$$= \sum_{(x^n, y^n)} p(x^n, y^n) \sum_{\substack{x^{n'} \neq x^n \\ (x^{n'}, y^n) \in A_\epsilon^{(n)}}} P\left[f_1(x^{n'}) = f_1(x^n)\right]$$

$$= \sum_{(x^n, y^n)} p(x^n, y^n)\, 2^{-nR_1} \left|A_\epsilon^{(n)}(X|y^n)\right| \simeq 2^{-nR_1}\, 2^{nH(X|Y)} \to 0 \text{ if } R_1 > H(X|Y)$$

Όμοια $P(E_2) \to 0$ if $R_2 > H(Y|X)$

$P_{12} \to 0$ if $R_1 + R_2 > H(X,Y)$

Thus, $\exists\, (f_1^*, f_2^*)$ with prob of error $< 4\epsilon$

Converse for Slepian-Wolf

Let f_1, f_2, g fixed. Let $I_0 = f_1(X^n)$, $J_0 = f_2(Y^n)$

Then

$$H(X^n, Y^n \mid I_0, J_0) \le P_e^{(n)}(\log|X| + \log|Y|) + 1 = n\epsilon_n, \quad \epsilon_n \to 0 \text{ as } n \to \infty$$

$$H(X^n \mid Y^n, I_0, J_0) \le n\epsilon_n$$

$$H(Y^n \mid X^n, I_0, J_0) \le n\epsilon_n$$

$$n(R_1+R_2) \ge H(I_0, J_0) = I(X^n, Y^n; I_0, J_0) + H(I_0, J_0 \mid X^n, Y^n)$$

$$= I(X^n, Y^n; I_0, J_0) = H(X^n, Y^n) - H(X^n, Y^n \mid I_0, J_0) \ge H(X^n, Y^n) - n\epsilon_n = nH(X,Y) - n\epsilon_n$$

Αντίστοιχα $nR_1 \ge H(I_0) \ge H(I_0 \mid Y^n) = I(X^n; I_0 \mid Y^n) + H(I_0 \mid X^n, Y^n) = I(X^n; I_0 \mid Y^n)$

$$= H(X^n \mid Y^n) - H(X^n \mid I_0, J_0, Y^n) \ge H(X^n \mid Y^n) - n\epsilon_n = nH(X \mid Y) - n\epsilon_n$$

και $nR_2 \ge nH(Y \mid X) - n\epsilon_n$

S-W with many sources

Θεώρημα. $(X_{1i}, X_{2i}, \dots, X_{mi})$ iid $\sim p(x_1, \dots, x_m)$. Then, achievable by distributed s.c.

$$\forall S \subseteq \{1, \dots, m\}, \quad R(S) = \sum_{i \in S} R_i > H(X(S) \mid X(S^c)), \quad \text{για } X(S) = \{X_j : j \in S\}.$$