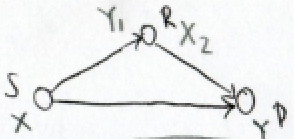
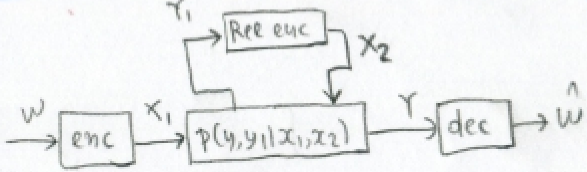
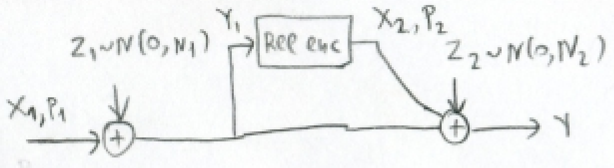
Relay Channel.



$$\boxed{(\mathcal{X}_1 \times \mathcal{X}_2,\ p(y, y_1 | x_1, x_2),\ \mathcal{Y} \times \mathcal{Y}_1)}$$



Relay channel



Degraded Gaussian channel.

Όπως θα δούμε.

$$C^* = \max_{0 \le \alpha \le 1} \min \left\{ C\left(\frac{P_1 + P_2 + 2\sqrt{\bar{\alpha} P_1 P_2}}{N_1 + N_2} \right), C\left(\frac{\alpha P_1}{N_1} \right) \right\}$$

Relay, using Y_1, recovers X_1 perfectly and then x_2 and x_1 cooperate coherently in the next block to resolve the remaining ambiguity of X_1 in Y.
Also, fresh X_1 info is sent.
And this iterates...

(M, n) code for relay:

$$\mathcal{M} = \{1, 2, \dots, M\}$$

enc: $X_1 : \mathcal{M} \to \mathcal{X}_1^n$

a set of relay functions $\{f_i\}_{i=1}^n$ such that

$$x_{2i} = f_i(Y_{1,1}, Y_{1,2}, \dots, Y_{1,i-1}), \quad 1 \le i \le n. \quad \left(\begin{array}{l} \text{non-anticipatory} \\ \text{relay nature} \end{array} \right)$$

dec: $g : \mathcal{Y}^n \to \mathcal{M}$.

$$p(w, \underline{x}_1, \underline{x}_2, \underline{y}, \underline{y}_1) = p(w) \prod_{i=1}^n p(x_{1i} | w)\, p(x_{2i} | y_{1,1}, y_{1,2}, \dots, y_{1,i-1})\, p(y_i, y_{1i} | x_{1i}, x_{2i}) \quad (5)$$

$\lambda(w) = \Pr\{ g(Y) \neq w \mid W = w \}$ conditional probability of error.

$$P_n(e) = \frac{1}{M} \sum_w \lambda(w)$$

$$\lambda_n = \max_w \lambda(w).$$

$$R = \frac{1}{n} \log M \text{ bits/transmission}$$

Degraded relay channel

$$p(y, y_1 | x_1, x_2) = p(y_1 | x_1, x_2) \cdot p(y | y_1, x_2)$$

$$X_1 \to (X_2, Y_1) \to Y$$

Theorem 1: Capacity C of the degraded relay channel is

$$C = \sup_{p(x_1, x_2)} \min \{ I(X_1, X_2; Y), I(X_1; Y_1 | X_2) \}$$

Comments: rate $I(X_1, X_2; Y)$ can be achieved for any $p(x_1, x_2)$. However, this can be done only if the relay can perfectly reconstruct X_1, ie, $R < I(X_1; Y | X_2)$.

We shall achieve capacity using a different encoding scheme at the relay.

Achievability of C.

- sequence of B blocks, n symbols each
- " of $B-1$ messages $w_i \in [1, 2^{nR}]$, $i = 0, \dots, B-1$, will be sent over nB transmissions. (as $B \to \infty$, for fixed n, rate $\frac{R(B-1)}{B} \to R$).

For each n-block $b = 1, \dots, B$, we use the same doubly index set of codewords.

$$\mathcal{C} = \{ \underline{x}_1(w_b | s_b), \underline{x}_2(s_b) \}, \quad w_b \in [1, 2^{nR}], \quad s_b \in [1, 2^{nR_0}]$$

$$\underline{x}_1(\cdot|\cdot) \in \mathcal{X}_1^n, \quad \underline{x}_2(\cdot) \in \mathcal{X}_2^n.$$

We shall need the partition

$$\mathcal{S} = \{ S_1, \dots, S_{2^{nR_0}} \} \text{ of } \mathcal{M} = \{1, \dots, 2^{nR}\}$$

into 2^{nR_0} cells, $S_i \cap S_j = \emptyset$, $\cup S_i = \mathcal{M}$.

- The partition $\mathcal{S}$ will allow us to send information to the receiver using Slepian-Wolf.
- The choice of $\mathcal{C}$ and $\mathcal{S}$ will be random.

Distribution of $\underline{x}_1(w_b | s_b)$ and $\underline{x}_2(s_b)$.

- Generate 2^{nR_0} iid $\sim p(x)$ $\underline{x}_2$ codewords with
$$p(\underline{x}_2) = \prod_{i=1}^{n} p(x_{2i}).$$
- For each $\underline{x}_2(s)$ generate 2^{nR} conditionally independent
$$\underline{x}_1(w|s), \quad p(\underline{x}_1(w|s)) = \prod_{i=1}^{n} p(x_{1i} | x_{2i}(s))$$
- Each $w \in [1, 2^{nR}]$ is assigned randomly and uniformly to a bin from $1, \dots, 2^{nR_0}$.

Assume a fixed code. $\boxed{S_i \text{ is the bin of } w_{i-1}, \forall i}$

We pick up the story at end of block $i-1$. We assume that

- $Rx\ \underline{y}$ knows w_{i-2} and s_{i-1} (previous message and his bin)
- Relay $\underline{y}_1$ knows w_{i-1} (current message)

With a good code, at end of block i:

- Rx will know (w_{i-1}, s_i)
- Relay will know w_i

Thus, the info state (w_{i-1}, s_i) at Rx and w_i at the relay propagate forward.

Transmission in block i: $\underline{x}_1(w_i | s_i)$, $\underline{x}_2(s_i)$

Received signals in block i: $\underline{Y}_1(i)$, $\underline{Y}(i)$.

Computation at end of block i.

① **Relay**: Using $\underline{Y}_1(i)$ it computes w_i. How?

Relay knows w_{i-1} and thus s_i and $\underline{X}_2(s_i)$.

$\underline{X}_1(w_i | s_i)$ is constructed by $\underline{X}_2(s_i)$.

Relay searches for w: $(X_1(w|s_i), X_2(s_i), Y_1) \in A_\epsilon^{n}$.

Remember degraded broadcast $(X(i,j), U(j), Y_1)$.

If $\boxed{R < I(X_1;Y_1|X_2)}$, then $\exists$ one w such that $(X_1(w|s_{i-1}), X_2(i-1), Y_1) \in A_\epsilon^{(n)}$ and this is $w = W_{i-1}$.

② (a) The receiver declares $\hat{s}_i = s$ if there is only one s such that $(\underline{x}_2(s), \underline{Y}(i)) \in A_\epsilon^{(n)}$. If $\boxed{R_0 < I(X_2;Y)}$ decoding of s_i can be done with arbitrary accuracy.

(b) The receiver calculates the ambiguity set $\mathcal{L}(\underline{Y}(i-1))$ (previous block)

$$\mathcal{L}(\underline{Y}(i-1)) = \{ w \in W : (\underline{x}_1(w_i|s_{i-1}), x_2(s_{i-1}), \underline{y}(i-1)) \in A_\epsilon^{(n)}$$

ⓒ The receiver intersects $\mathcal{L}(\underline{Y}(i-1))$ and bin S_{s_i}.

If

$$\boxed{R < I(X_1;Y|X_2) + R_0}$$

then, with high probability, then $\mathcal{L}(\underline{Y}(i-1)) \cap S_{s_i}$ will contain one $x(\hat{w}|s_i)$ and $\hat{w} = w_{i-1}$.

Combining $R_0 < I(X_2;Y)$ and $R < I(X_1;Y|X_2) + R_0$, we get

$$\boxed{R < I(X_1, X_2; Y)}$$

Calculation of Probability of error.

We declare an error in block i if one or more of the following events occurs.

$E_{0i}: (\underline{x}_1(w_i|s_i), \underline{x}_2(s_i), \underline{y}_1(i), \underline{y}(i)) \notin A_\epsilon^{(n)}$

E_{1i}: in decoding step 1: $\exists \tilde{w} \neq w_i: (\underline{x}_1(\tilde{w}|s_i), \underline{x}_2(s_i), \underline{Y}_1(i)) \in A_\epsilon^{(n)}$

E_{2i}: " " 2: $\exists \tilde{s} \neq s_i: (\underline{x}_2(\tilde{s}), \underline{y}(i)) \in A_\epsilon^{(n)}$

E_{3i}: decoding step 3 fails: Let $E_{3i} = E'_{3i} \cup E''_{3i}$, where

$E'_{3i}: w_{i-1} \notin S_{s_i} \cap \mathcal{L}((\underline{y}(i-1))$

$E''_{3i}: \exists \tilde{w} \neq w_{i-1}: \tilde{w} \in S_{s_i} \cap \mathcal{L}(\underline{y}(i-1))$

Let $F_i = \{\hat{w}_i \neq w_i \text{ or } \hat{w}_{i-1} \neq w_{i-1} \text{ or } \hat{s}_i \neq s_i\} = \bigcup_{k=0}^{3} E_{ki}$.

It can be shown that

$$P(E_{0i}|F^c_{i-1}) \leq \frac{\epsilon}{4B}$$

$$P(E_{1i} \cap E^c_{0i}|F^c_{i-1}) \leq \frac{\epsilon}{4B}$$

$$P(E_{2i} \cap E^c_{0i}|F^c_{i-1}) \leq \frac{\epsilon}{4B}$$

Lemma: If $R < I(X_1;Y|X_2) + R_0 - 7\epsilon$, then for sufficiently large n

$$P(E_{3i} \cap E^c_{2i} \cap E^c_{0i} | F^c_{i-1}) \leq \frac{\epsilon}{4B}.$$

Proof: First, we bound $E\{\|\mathcal{L}(\underline{Y}(i-1))\| \,|\, F^c_{i-1}\}$, $\|\mathcal{L}\|$: cardinality of $\mathcal{L}$

Let

$$\psi(w|\underline{y}(i-1)) = \begin{cases} 1 & (\underline{x}_1(w|s_{i-1}), \underline{x}_2(s_{i-1}), \underline{y}(i-1)) \in A_\epsilon^{(n)} \\ 0, & \text{otherwise} \end{cases}$$

$$\|\mathcal{L}(\underline{y}(i-1))\| = \sum_w \psi(w|\underline{y}(i-1))$$

and

$$E\{\|\mathcal{L}(\underline{y}(i-1))\| \,|\, F^c_{i-1}\} = E\{\psi(w_{i-1}|\underline{y}(i-1)) | F^c_{i-1}\} + \sum_{w \neq w_{i-1}} E\{\psi(w|\underline{y}(i-1)) | F^c_{i-1}\}$$

For $w \neq w_{i-1}$: $E\{\psi(w|\underline{y}(i-1)) | F^c_{i-1}\} \simeq 2^{-nI(X_1;Y|X_2)}$

Therefore: $E\{\|\mathcal{L}(\underline{y}(i-1))\| \,|\, F^c_{i-1}\} \leq 1 + (2^{nR} - 1)\, 2^{-nI(X_1;Y|X_2)} \leq 1 + 2^{n(R - I(X_1;Y|X_2))}$

$-$ $F^c_{i-1} \Rightarrow w_{i-1} \in \mathcal{L}(\underline{y}(i-1))$.

$E^c_{2i} \Rightarrow \hat{s}_i = s_i \Rightarrow w_{i-1} \in S_{\hat{s}_i}$. Thus

$$P(E'_{3i} \cap E^c_{2i} \cap E^c_{0i} | F^c_{i-1}) = 0.$$

Thus.

$$P(E_{3i} \cap E_{2i}^c \cap E_{0i}^c \mid F_{i-1}^c) = P(E_{3i}'' \cap E_{2i}^c \cap E_{0i}^c \mid F_{i-1}^c)$$

$$\leq P\{\exists \ w \neq W_{i-1} \text{ such that}$$

$$w \in \mathcal{L}(\underline{y}(i-1)) \cap S_{s_i} \mid F_{i-1}^c\}$$

$$\leq E\{\|\mathcal{L}(\underline{y}(i-1))\| \, 2^{-nR_0} \mid F_{i-1}^c\}$$

$$\leq 2^{-nR_0}\left(1 + 2^{n(R - I(X_1;Y|X_2))}\right).$$

If $R_0 > R - I(X_1;Y|X_2)$, then

$$P(E_{3i} \cap E_{2i}^c \cap E_{0i}^c \mid F_{i-1}^c) \leq \frac{\epsilon}{4B}.$$

We also have that $R_0 < I(X_2;Y)$. Thus, for

$$R < (X_1;Y|X_2) + I(X_2;Y) = I(X_1,X_2;Y)$$

Seminar 6

A few words about codebooks in degraded Gaussian relays

$\underline{\tilde{X}}_1(w)$ iid $\sim N_n(0, \alpha P_1 I_n)$, $w \in [1, 2^{nR}]$

$\underline{X}_2(s)$ iid $\sim N_n(0, P_2 I_n)$, $s \in [1, 2^{nR_0}]$.

$$\underline{x}_1(w|s) = \underline{\tilde{x}}_1(w) + \sqrt{\frac{\bar{\alpha} P_1}{P_2}}\, \underline{x}_2(s)$$

$\underline{x}_2(s)$

Άρα, στο επόμενο block, ο source και ο relay send

coherently $\underline{x}_2(s)$ and $\underline{x}_1(s)$.

The source also sends new information $\underline{\tilde{x}}_1(w)$

independent of $\underline{x}_2$