

# OUTAGE CAPACITY OF A COOPERATIVE SCHEME WITH BINARY INPUT AND A SIMPLE RELAY

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## ABSTRACT

Cooperative communications is a rapidly evolving research area. Most of the cooperative protocols that have appeared in the literature assume slow flat fading channels and Gaussian codebooks. In many cases the relays must fully decode their input. It is well known that cooperation is most effective at low SNR where binary input is optimal. Furthermore, energy and cost effectiveness make simple relays most attractive. Motivated by these two facts, we consider a half-duplex orthogonal cooperation protocol with binary input and relays that simply forward their symbol-by-symbol decisions to the destination which performs algebraic decoding; we call it *Demodulate-and-Forward (DmF)*. We assume independent slow Rayleigh flat fading channels with full channel state information (CSI) at the destination and compute an upper bound for the outage capacity of the DmF protocol. For low SNR and small outage probability, we derive a simple approximation to this bound. For comparison purposes, we compute the outage capacity of direct binary transmission and a simple low-SNR small-outage-probability approximation. We observe that for very small outage probability the DmF protocol significantly outperforms direct transmission. However, for (relatively) high outage probability, the opposite may happen.

**Index Terms**— Cooperation diversity, outage capacity.

## I. INTRODUCTION

Cooperative diversity is a concept that has recently attracted significant research interest. Its origin can be found in the information theoretic relay channel model studied by Van der Meulen [1] and Cover and El Gamal [2] more than thirty years ago. The work of Sendonaris *et al.* [3] and Laneman *et al.* [4] renewed the interest in the concept of cooperation and has led to the development of many cooperation protocols like the amplify-and-forward, decode-and-forward and their selective and incremental variations [4], and coded diversity protocols [5]. Outage probability, diversity order, and diversity-multiplexing tradeoff have been the main performance measures used in the study of the above protocols [4], [6]. We note that most of the above protocols assume Gaussian codebooks.

It is well established that cooperation is most effective at low SNR where binary input is optimal [7]. Furthermore,

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energy consumption and manufacturing costs make simple relays most attractive. Thus, cooperation protocols with binary input and simple relays, that is, relays that do *not* have to fully decode their input but simply forward to the destination their symbol-by-symbol decisions, are of great importance. However, no information theoretic study of such a scheme has appeared in the literature.

In this work, we consider an orthogonal cooperative protocol with binary input and relays that forward their symbol-by-symbol decisions to the destination, which performs algebraic decoding;<sup>1</sup> we call it *demodulate-and-forward (DmF)*.<sup>2</sup> First, we compute the capacity of a channel with binary input and two independent looks at the output. Using this capacity expression and assuming independent slow Rayleigh flat fading channels, we compute an upper bound of the outage capacity of DmF. Then, we derive a simple approximation to this bound for the important case of small outage probability and low SNR. For comparison purposes, we also compute the outage capacity of direct binary transmission and its low-SNR small-outage-probability approximation. Using a mixture of analysis and simulations, we conclude that, for very small outage probability, the DmF protocol attains significantly higher outage capacity than direct transmission over a wide range of SNR values. However, for moderate and high outage probability the opposite may happen.

## II. THE DMF PROTOCOL

We consider the relay channel model depicted in Fig. 1 and assume independent slow Rayleigh flat fading channels. Thus,  $h_{sd}$ ,  $h_{sr}$ , and  $h_{rd}$  are independent complex-valued circular Gaussian random variables with variances  $\sigma_{sd}^2$ ,  $\sigma_{sr}^2$ , and  $\sigma_{rd}^2$ , respectively. The transmit power at both the source and the relay is  $P$ . The input of the relay and the destination is corrupted by white circular Gaussian noise with variance  $N_0$ . The transmit SNR is denoted by  $\Gamma := \frac{P}{N_0}$ . The instantaneous receive SNRs  $\gamma_{sd}$ ,  $\gamma_{sr}$ , and  $\gamma_{rd}$  are exponential

<sup>1</sup>Derivation of closed-form expressions for the cases where the destination performs soft-decision decoding seems very difficult.

<sup>2</sup>The DmF protocol has been considered in the uncoded case in [8].

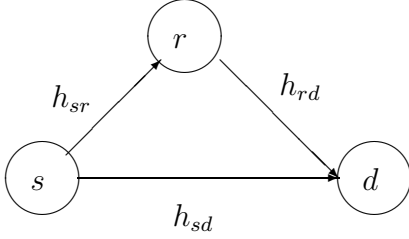


Fig. 1. Relay channel model

random variables with means  $\Gamma_{sd} := \sigma_{sd}^2 \Gamma$ ,  $\Gamma_{sr} := \sigma_{sr}^2 \Gamma$ , and  $\Gamma_{rd} := \sigma_{rd}^2 \Gamma$ , respectively.

The DmF protocol assumes half-duplex relays and orthogonal transmissions. By assuming orthogonality in time, the DmF protocol occupies two time slots and operates as follows. During the first time slot, the source emits a length- $N$  binary codeword to the destination overheard by the relay. The signals received at the destination and relay at time instants  $i = 1, \dots, N$  are given by

$$y_{1i} = h_{sd} x_i + w_{1i},$$

$$y_{Ri} = h_{sr} x_i + w_{Ri}.$$

The relay uses perfect instantaneous CSI and performs symbol-by-symbol coherent demodulation to decide in favor of

$$\hat{x}_i = \text{sgn}(\text{Re}(h_{sr}^* y_{Ri}))$$

with error probability  $\rho_{sr} = Q(\sqrt{2\gamma_{sr}})$ . During the second time slot, the relay forwards its decisions to the destination which receives

$$y_{2i} = h_{rd} \hat{x}_i + w_{2i}.$$

The destination performs algebraic decoding, that is, at first demodulates its input to obtain

$$\hat{x}_{1i} = \text{sgn}(\text{Re}(h_{sd}^* y_{1i})), \quad \hat{x}_{2i} = \text{sgn}(\text{Re}(h_{rd}^* y_{2i}))$$

with probability of error  $\rho_{sd} = Q(\sqrt{2\gamma_{sd}})$  and  $\rho_{rd} = Q(\sqrt{2\gamma_{rd}})$ , respectively, and then, using all error probabilities, performs full decoding.

The DmF protocol can be modeled by the structure of binary symmetric channels (BSCs) depicted in Fig. 2. Of course, the lower branch can be replaced by one BSC with crossover probability  $\rho_{sr} + \rho_{rd} - 2\rho_{sr}\rho_{rd}$ .

In the sequel, we perform an information theoretic study of the DmF protocol. A function that will prove very useful towards this purpose is

$$\mathcal{H}(\rho) := -\rho \log_2 \rho - (1-\rho) \log_2 (1-\rho), \quad \rho \in \left[0, \frac{1}{2}\right]. \quad (1)$$

$\mathcal{H}(\rho)$  is concave and strictly increasing; its derivative  $\mathcal{H}'(\rho)$  is strictly decreasing. With  $\mathcal{H}_{\text{inv}}(x)$ ,  $x \in [0, 1]$ , we denote its inverse function. We recall that the capacity of a BSC with crossover probability  $\rho$  is  $C_\rho^{\text{BSC}} = 1 - \mathcal{H}(\rho)$ .

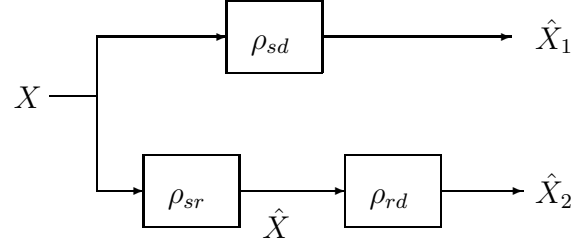


Fig. 2. Demodulate-and-Forward with algebraic decoding at the destination.

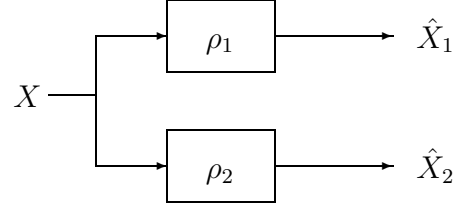


Fig. 3. Channel with binary input and two independent looks at the output.

### III. CHANNEL WITH BINARY INPUT AND TWO INDEPENDENT LOOKS AT THE OUTPUT

We consider the channel with binary input  $X$  and two independent looks at the output  $\hat{X}_1$  and  $\hat{X}_2$ , depicted in Fig. 3. The analysis of this channel is fundamental for the study of the DmF protocol in the next section. Its capacity is given by the following proposition.

*Proposition 1:* The capacity of a channel with binary input  $X$ , two independent looks at the output  $\hat{X}_1$  and  $\hat{X}_2$ , and crossover probabilities  $\rho_1$  and  $\rho_2$ , respectively, is achieved for uniform input and is given by

$$C_{\rho_1, \rho_2} = 1 + \mathcal{H}(\rho_1 + \rho_2 - 2\rho_1\rho_2) - \mathcal{H}(\rho_1) - \mathcal{H}(\rho_2). \quad (2)$$

*Proof:* Due to space limitation, we refer the interested reader to [9].  $\square$

In the next section, we shall make use of the following Lemma.

*Lemma 1:* Let  $\rho_1, \rho_2 \in [0, \frac{1}{2}]$ . Then

$$\mathcal{H}(2\rho_1\rho_2) \leq \mathcal{H}(\rho_1) + \mathcal{H}(\rho_2) - \mathcal{H}(\rho_1 + \rho_2 - 2\rho_1\rho_2). \quad (3)$$

*Proof:* The proof is based on the inequality

$$f(y) \leq f(x) + f'(x)(y-x) \quad (4)$$

which holds for any concave function  $f$  with derivative  $f'$  [10, p. 70] and the fact that  $\mathcal{H}'(\rho)$  is decreasing for  $\rho \in [0, \frac{1}{2}]$ . Let us assume that  $\rho_1 \leq \rho_2$ . Applying (4) twice, we obtain

$$\mathcal{H}(2\rho_1\rho_2) \leq \mathcal{H}(\rho_1) + \mathcal{H}'(\rho_1) \underbrace{(2\rho_1\rho_2 - \rho_1)}_{\alpha},$$

$$\mathcal{H}(\rho_1 + \rho_2 - 2\rho_1\rho_2) \leq \mathcal{H}(\rho_2) + \mathcal{H}'(\rho_2)(\rho_1 - 2\rho_1\rho_2). \quad (5)$$

Since  $\rho_1 \leq \rho_2$ , we have  $\mathcal{H}'(\rho_1) - \mathcal{H}'(\rho_2) \geq 0$ . Furthermore, for any  $\rho_1, \rho_2 \in [0, \frac{1}{2}]$ , it can be shown that  $\alpha \leq 0$ . Inequality (3) is proved if we add the inequalities in (5) and observe that  $\alpha(\mathcal{H}'(\rho_1) - \mathcal{H}'(\rho_2))$  is nonpositive.  $\square$

#### IV. INFORMATION THEORETIC ANALYSIS OF THE DMF PROTOCOL

We assume independent slow Rayleigh flat fading channels and algebraic decoding at the destination. If we set  $\rho_1 := \rho_{sd}$  and  $\rho_2 := \rho_{sr} + \rho_{rd} - 2\rho_{sr}\rho_{rd}$ , then the DmF scheme of Fig. 2 is equivalent to the channel with two independent looks of Fig. 3 besides the fact that a channel use of the DmF protocol needs twice the time needed by a channel use of the channel with two independent looks. This occurs because the transmission of one codeword with the DmF protocol needs two time slots. Thus, for fixed  $\gamma_{sr}, \gamma_{sd}$ , and  $\gamma_{rd}$ , the *instantaneous* capacity of the DmF scheme is

$$C_{\rho_1, \rho_2}^{DmF} = \frac{1}{2} C_{\rho_1, \rho_2}. \quad (6)$$

The outage probability of the DmF protocol for rate  $R \in [0, \frac{1}{2}]$  is defined as

$$P_{\text{out}}^{DmF}(R) := P[C_{\rho_1, \rho_2}^{DmF} < R]. \quad (7)$$

Since  $\rho_{sr}, \rho_{rd} \in [0, 1/2]$ ,

$$\rho_2 \geq \max\{\rho_{sr}, \rho_{rd}\} =: \rho_{srd}. \quad (8)$$

We define  $\gamma_{srd} := \min\{\gamma_{sr}, \gamma_{rd}\}$ .  $\gamma_{srd}$  is an exponential random variable with mean

$$\Gamma_{srd} := \frac{\Gamma_{sr}\Gamma_{rd}}{\Gamma_{sr} + \Gamma_{rd}} = \frac{\sigma_{sr}^2\sigma_{rd}^2}{\sigma_{sr}^2 + \sigma_{rd}^2} \Gamma. \quad (9)$$

The outage probability of the DmF protocol is upper bounded as follows:<sup>3</sup>

$$\begin{aligned} P_{\text{out}}^{DmF}(R) &= P[1 + \mathcal{H}(\rho_1 + \rho_2 - 2\rho_1\rho_2) - \mathcal{H}(\rho_1) - \mathcal{H}(\rho_2) < 2R] \\ &\stackrel{(3)}{\geq} P[\mathcal{H}(2\rho_1\rho_2) > 1 - 2R] \\ &= P[\rho_1\rho_2 > \underbrace{\frac{1}{2} \mathcal{H}_{\text{inv}}(1 - 2R)}_{c_1(R)}] \\ &\stackrel{(8)}{\geq} P[\rho_{sd}\rho_{srd} > c_1(R)] \\ &= P[Q(\sqrt{2\gamma_{sd}})Q(\sqrt{2\gamma_{srd}}) > c_1(R)] \\ &\stackrel{(a)}{\geq} P[Q^2(\sqrt{\gamma_{sd} + \gamma_{srd}}) > c_1(R)] \\ &= P[\gamma_{sd} + \gamma_{srd} < \underbrace{Q_{\text{inv}}^2(\sqrt{c_1(R)})}_{c_2(R)}] \\ &= 1 - \frac{1}{\Gamma_{sd} - \Gamma_{srd}} \left( \Gamma_{sd} e^{-\frac{c_2(R)}{\Gamma_{sd}}} - \Gamma_{srd} e^{-\frac{c_2(R)}{\Gamma_{srd}}} \right) \\ &\approx \frac{c_2^2(R)}{2\Gamma_{sd}\Gamma_{srd}} \end{aligned}$$

<sup>3</sup>Due to space limitation, we ignore the case  $\Gamma_{srd} = \Gamma_{sd}$ .

where inequality (a) can be proved using the convexity of  $\ln Q(\sqrt{x})$  and the approximation at the last line is accurate for sufficiently small outage probability. To compute an upper bound for the  $\epsilon$ -outage capacity of the DmF protocol  $R_\epsilon^{DmF}$ , we replace  $P_{\text{out}}^{DmF}(R)$  and  $R$  by  $\epsilon$  and  $R_\epsilon^{DmF}$ , respectively, and obtain

$$\frac{c_2^2(R_\epsilon^{DmF})}{2\Gamma_{sd}\Gamma_{srd}} \lesssim \epsilon$$

which is equivalent to

$$R_\epsilon^{DmF} \approx \frac{1}{2} \left( 1 - \mathcal{H} \left( 2Q^2(\sqrt{2\Gamma_{sd}\Gamma_{srd}\epsilon}) \right) \right). \quad (10)$$

For small  $\epsilon$ ,  $\Gamma_{sd}$ , and  $\Gamma_{srd}$ , using Taylor expansions, we approximate the right-hand side of (10) and obtain

$$R_\epsilon^{DmF} \approx \frac{2 \log_2 e}{\pi} \sqrt{2\Gamma_{sd}\Gamma_{srd}\epsilon}. \quad (11)$$

We observe that the (approximate) upper bound for the outage capacity of the DmF protocol is proportional to  $\sqrt{\epsilon}$ , implying that, if this bound is tight, then the DmF protocol offers second-order diversity [7].

#### IV-A. Outage capacity of direct transmission

For comparison purposes, we compute the  $\epsilon$ -outage capacity of direct transmission by assuming a slow Rayleigh flat fading channel  $h_{sd}$  with binary input and algebraic decoding. This channel can be modeled as a BSC with crossover probability  $\rho'_{sd} = Q(\sqrt{2\gamma'_{sd}})$ . In order to perform a fair comparison with the DmF protocol, we assume that, in direct transmission, the transmitted source power is  $2P$ . Thus,  $\gamma'_{sd}$  is an exponential random variable with mean  $\Gamma'_{sd} = 2\Gamma_{sd}$ .

The capacity of this channel is  $C_{\rho'_{sd}}^{BSC} = 1 - \mathcal{H}(\rho'_{sd})$  and the outage probability for rate  $R \in [0, 1]$  is

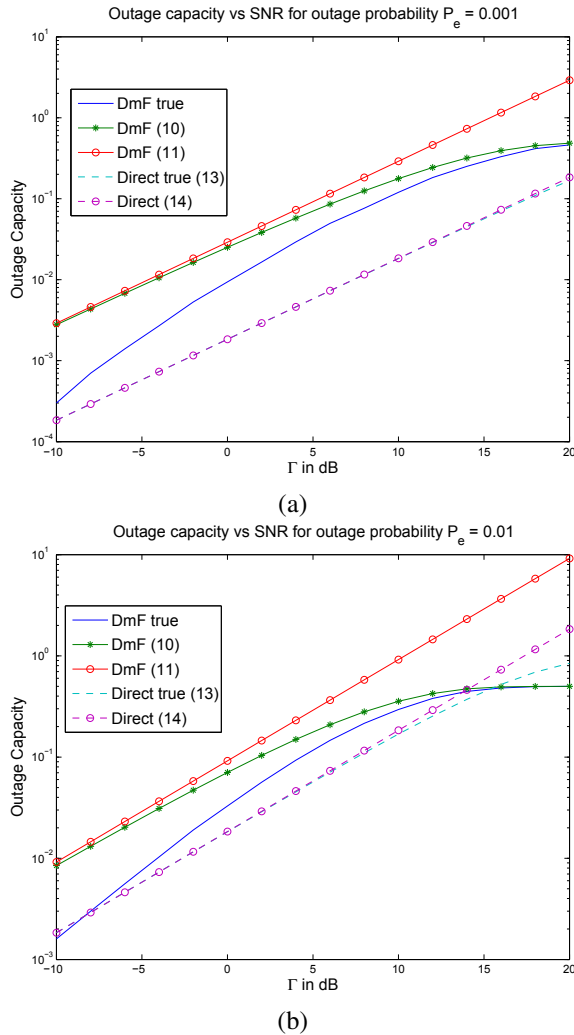
$$\begin{aligned} P_{\text{out}}^{BSC}(R) &= P[C_{\rho'_{sd}}^{BSC} < R] = P[1 - \mathcal{H}(\rho'_{sd}) < R] \\ &= P[\rho'_{sd} > \mathcal{H}_{\text{inv}}(1 - R)] \\ &= P[Q(\sqrt{2\gamma'_{sd}}) > \mathcal{H}_{\text{inv}}(1 - R)] \\ &= P[\sqrt{2\gamma'_{sd}} < Q_{\text{inv}}(\mathcal{H}_{\text{inv}}(1 - R))] \\ &= P[\gamma'_{sd} < \frac{1}{2} Q_{\text{inv}}^2(\mathcal{H}_{\text{inv}}(1 - R))] \\ &= 1 - e^{-\frac{Q_{\text{inv}}^2(\mathcal{H}_{\text{inv}}(1 - R))}{2\Gamma_{sd}}}. \end{aligned} \quad (12)$$

To find the  $\epsilon$ -outage capacity of direct transmission, we solve for  $R_\epsilon^{BSC}$  in

$$P_{\text{out}}^{BSC}(R_\epsilon^{BSC}) = \epsilon$$

and obtain

$$R_\epsilon^{BSC} = 1 - \mathcal{H} \left( Q \left( \sqrt{-2\Gamma'_{sd} \ln(1 - \epsilon)} \right) \right). \quad (13)$$



**Fig. 4.** Experimentally computed true outage capacity of the DmF protocol, upper bound (10), approximate bound (11), true outage capacity of direct transmission (13) and its approximation (14), for outage probability  $\epsilon = 0.001$  (a) and  $\epsilon = 0.01$  (b).

For small  $\epsilon$  and  $\Gamma'_{sd}$ , using Taylor expansions, we obtain

$$R_{\epsilon}^{BSC} \approx \frac{2 \log_2 e}{\pi} \Gamma'_{sd} \epsilon. \quad (14)$$

We observe that the outage capacity of the direct transmission is (approximately) proportional to  $\Gamma\epsilon$ , implying, non-surprisingly, first-order diversity.

## V. NUMERICAL RESULTS

As an illustration of our theoretical results, we consider a simple scenario where  $\Gamma_{sd} = \Gamma_{sr} = \Gamma_{rd} = \Gamma$ . In this case,  $\Gamma'_{sd} = \frac{\Gamma}{2}$ . In Fig. 4, we plot the experimentally computed true outage capacity of the DmF protocol, its

upper bound (10), and its approximation (11) as well as the true outage capacity of direct transmission (13) and its approximation (14), for outage probability 0.001 and 0.01. In Fig. 4a, we observe that for outage probability  $\epsilon = 0.001$  the DmF protocol offers significantly higher outage capacity than direct transmission over a wide range of SNR values. For higher outage probability, the DmF protocol is superior to direct transmission at moderate SNR and inferior at low and high SNR (see Fig. 4b). For high outage probability (not shown in the figure), direct transmission always outperforms DmF. Finally, we observe that bound (10) is tight for high SNR while approximations (14) and (11) are accurate for small  $\epsilon$  and  $\Gamma$ .

## VI. CONCLUSION

We considered an orthogonal cooperation protocol with binary input, relays that forward their symbol-by-symbol decisions and algebraic decoding at the destination. We computed an upper bound for the outage capacity of this protocol and a simple approximation to this bound. Comparison of the outage capacity of DmF with that of direct transmission reveals that the DmF cooperative protocol is most effective at very low outage probability.

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