Binary Transmissions over Additive Gaussian Noise:
A Closed-Form Expression for the Channel Capacity

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Abstract — We derive a closed-form expression for the capacity of the additive-Gaussian-noise memoryless channel with binary input. The capacity expression that we obtain contains an infinite series. Truncation of the series -for computation reasons- at an odd or even number of terms results in a lower or upper, respectively, bound on the channel capacity; we prove that the tightness of the bound increases monotonically with the number of terms that are included in the truncated series. Numerical results confirm that, as long as at least the first six terms of the series are calculated, the corresponding lower or upper bound becomes indistinguishable from the exact channel capacity. Therefore, with small computational complexity we obtain accurately the capacity of the Gaussian channel with binary input.

Index Terms — Channel capacity, decoding, discrete-input channels, encoding, information rates, Gaussian channels, Gaussian noise.

I. INTRODUCTION

The quality of a communication system is measured by the information transmission rate from the source to the destination. The overall transmission rate is determined by several external (e.g., physical medium, propagation environment) and design (e.g., compression, modulation) factors. In digital communication systems, such a design factor is the selection of the channel coding/decoding scheme that ensures communication with small probability of error between the source and the destination. Provided that the probability of error can be made arbitrarily small, the information transmission rate of a digital communication system is proportional to the channel coding rate, hence the latter should be selected as high as possible subject to the arbitrarily-small-error-probability constraint. The maximum possible coding rate for reliable communication depends on the transmitter, communication channel, and receiver characteristics and is called the channel capacity.

If the channel is memoryless, then its capacity is equal to the maximum mutual information between the coded input and output of the channel [1]. The maximization is performed over all possible coded input probability density functions (pdf’s) allowed by the encoder. Therefore, the channel capacity depends strictly on the (i) physical channel characteristics and (ii) transmitter (in fact, channel coder) specifics. We note that the channel input distribution is defined over the allowable input alphabet. Along these lines, the channel capacity of several input-alphabet/physical-channel pairs has been evaluated in the past. Primary examples are the binary symmetric channel, discrete memoryless channel, and Gaussian channel [2]. The latter is perhaps the most popular model; the input alphabet -usually with an average input power constraint enforced- is the entire real (or complex) field and Gaussian noise is added during transmission. The Gaussian channel is a practical model for many real channels, including radio and satellite links, because it represents the cumulative effect of a large number of small random effects in a communication system [2]. In practice, the capacity of the Gaussian channel is usually viewed as a benchmark of the quality of a communication system.

Although the capacity of the Gaussian channel is indeed a tight upper bound on the rate of any possible coding scheme, this bound becomes loose when the coded input alphabet is restricted to a proper subset of the real field. In fact, most of the coding methods that have been proposed until today produce binary coded symbols suitable for transmission with conventional modulation schemes. Hence, the capacity of the additive-Gaussian-noise (AGN) channel with binary input becomes a particularly important metric on the performance of practical communication systems.

By definition, the binary-input AGN channel capacity is given in the form of an integral [1]. Therefore, the capacity curve can be obtained by common numerical integration methods. Such methods have been used in [3]-[7] to produce the curve. In particular, Monte-Carlo based techniques are utilized in [3] and [7] for the evaluation of the channel capacity integral. However, due to the significance of the binary-input AGN channel capacity, more advanced numerical techniques tailored to the problem under consideration have appeared in the literature. Prime examples include numerical algorithms for the calculation of the capacity developed in [8]-[11]. On the other hand, in an attempt to obtain a closed-form expression for the capacity, new capacity formulas have been derived in [12], [13]. Although these formulas are significantly different than the original capacity definition [1], they still contain an integral which is to be numerically evaluated. Therefore, the capacity curves presented in [10], [13] can only be obtained by numerical integration. As a result, the problem of obtaining a closed-form expression for the binary-input AGN channel capacity has often been reported as open in past literature (see
for example [7], [9], [11], [14]).

In this present work, we consider exactly the above problem and obtain the channel capacity for binary transmissions over additive Gaussian noise in a closed-form expression. As a side result, we prove that this maximum rate is achieved when the binary input distribution is uniform. As expected, the channel capacity is a monotonically increasing function of the constraint on the input power-to-noise-variance-ratio, usually referred to as input signal-to-noise-ratio (SNR). At the absence of any input SNR constraint (or, equivalently, when the constraint becomes infinite) the channel capacity becomes equal to 1, in contrast to the Gaussian channel with real/complex-valued inputs for which it becomes arbitrarily large (infinite). We note that this new capacity expression represents the maximum rate of coding schemes of practical interest for reliable communication. In other words, it indicates the error-free performance limit for common coding techniques, like Turbo [15] or LDPC [16] codes, etc.

The capacity expression that we obtain consists of an infinite series. Of course, computation limitations might impose restrictions on the number of terms that are included in the calculation of the capacity. We prove that when the infinite series is truncated at an odd number of terms it represents a lower bound on the capacity. Similarly, when it is truncated at an even number it becomes an upper bound. Both bounds become tighter as the number of summation terms increases. Therefore, for a large enough number of terms, the two bounds actually serve as accurate approximations/evaluations of the exact capacity.

The rest of this paper is organized as follows. In Section II we evaluate the exact channel capacity for binary transmissions over additive Gaussian noise. Lower and upper bounds on the capacity are obtained in Section III through appropriate truncation of the infinite series.

II. Binary Transmissions Over Additive Gaussian Noise: The Exact Channel Capacity

Let us denote by the random variables $X$ and $Y$ the coded input and output, respectively, of a discrete-time memoryless communication channel. The capacity of such a channel is given by

$$C \triangleq \max_{f_X \in \mathcal{F}} \{ I(X;Y) \}$$

where $I(X;Y)$ is the mutual information between $X$ and $Y$, $f_X$ denotes the pdf of the input random variable $X$, and $\mathcal{F}$ is the set of allowable input pdf’s. In general, the channel capacity $C$ depends on the set of allowable input pdf’s $\mathcal{F}$ as well as the relation that characterizes the connection between the input $X$ and output $Y$ expressed by the conditional pdf of the output given the input $f_{Y|X}$. The selection of $\mathcal{F}$ usually follows restrictions/constraints enforced by the available technology while the conditional pdf $f_{Y|X}$ depends on the propagation environment.

In this work we concentrate on the additive Gaussian noise channel which is described by

$$Y = X + N$$

where $N$ is a Gaussian random variable with variance $\sigma^2$ that represents additive noise. All three variables $X$, $N$, and $Y$ are considered to be real-valued, $X, N, Y \in \mathbb{R}$, and -without loss of generality- we assume that $N$ is zero-mean, i.e. $N \sim \mathcal{N}(0, \sigma^2)$. The channel input $X$ and noise $N$ are assumed independent from each other.

Let us first impose a maximum-input-power constraint $E \{X^2\} \leq p$. Then, the set of allowable input pdf’s is given by

$$\mathcal{F}_R(p) = \{ f_X : E \{X^2\} \leq p \}.$$

Maximization of $I(X;Y)$ over $\mathcal{F}_R(p)$ results in the popular capacity expression $C_R(p, \sigma) = \frac{1}{2} \log_2(1 + \frac{p}{\sigma^2})$. The capacity $C_R(p, \sigma)$ is achieved if and only if the channel input $X$ is zero-mean Gaussian with power $p$, i.e. $X \sim \mathcal{N}(0, p)$, hence the input alphabet is the entire set of real numbers $\mathbb{R}$. Since $C_R(p, \sigma)$ is a function of the input SNR constraint $\gamma \triangleq \frac{p}{\sigma^2}$, we rewrite it as

$$C_R(\gamma) = \frac{1}{2} \log_2(1 + \gamma).$$

However, as mentioned earlier, the coded symbols are binary for most of the existing channel coding schemes. As a result, the capacity $C_R(\gamma)$ becomes a loose upper bound on the coding rate of such schemes, provided that arbitrarily small probability of error is desired. To obtain an upper bound for binary transmissions, that is the capacity of the AGN channel with binary input, we need to identify first the set of allowable input distributions. If a maximum-input-power constraint $E \{X^2\} \leq p$ is imposed, then the set of allowable input pdf’s is given by

$$\mathcal{F}_B(p) = \{ f_X : f(x) = \rho \delta(x - \sqrt{p}) + (1 - \rho) \delta(x + \sqrt{p}), \rho \in [0,1], \rho' \leq p \} \subset \mathcal{F}_R(p),$$

where $\delta(x)$ denotes the Dirac delta function, and the corresponding channel capacity becomes

$$C_B(p, \sigma) = \max_{f_X \in \mathcal{F}_B(p)} \{ I(X;Y) \}.$$ (6)

Instead of evaluating $C_B(p, \sigma)$ directly, let us consider at this point a constant-input-power constraint $E \{X^2\} = p$ and the corresponding set of allowable input pdf’s

$$\mathcal{F}_B^*(p) = \{ f_X : f(x) = \rho \delta(x - \sqrt{p}) + (1 - \rho) \delta(x + \sqrt{p}), \rho \in [0,1] \}.$$ (7)

Following, we first derive the corresponding capacity

$$C_B^*(p, \sigma) = \max_{f_X \in \mathcal{F}_B^*(p)} \{ I(X;Y) \}$$

and subsequently show that $C_B(p, \sigma) = C_B^*(p, \sigma)$.

We begin our derivation by applying a few basic properties from information theory to the definition of $C_B^*(p, \sigma)$ and taking into account the Gaussian noise pdf.

$$C_B^*(p, \sigma) = \max_{f_X \in \mathcal{F}_B^*(p)} \{ I(X;Y) \}$$

$$= \max_{f_X \in \mathcal{F}_B^*(p)} \{ h(Y) - h(Y|X) \}$$

$$= \max_{f_X \in \mathcal{F}_B^*(p)} \{ h(Y) - h(N) \}$$

$$= \max_{f_X \in \mathcal{F}_B^*(p)} \{ h(Y) \} - \log_2(\sqrt{2\pi e\sigma^2}).$$

Since $f_X \in \mathcal{F}_B^*(p)$, the distribution of $Y$ consists of a Gaussian mixture parameterized in $\rho$,

$$f_Y(x) = \frac{\rho}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x-\sqrt{p})^2}{2\sigma^2}} + \frac{1-\rho}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x+\sqrt{p})^2}{2\sigma^2}},$$

which implies that the entropy $h(Y)$ is a function of $\rho$ as well. Therefore, to evaluate $C_B^*(p, \sigma)$ it is sufficient to (i) find the
value of $\rho \in [0, 1]$ that maximizes $h(Y)$ and (ii) evaluate $h(Y)$ for that particular optimal $\rho$ value.

Interestingly enough, maximization of $h(Y)$ over $\rho \in [0, 1]$ results in $\rho = \frac{1}{2}$, according to the following proposition. The proof is omitted due to lack of space.\footnote{The conclusion of Proposition 1 is stated in [6], [17], [18] without a proof.}

**Proposition 1** If the random variable $Y$ is distributed according to the density function $f_Y(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x-\gamma)^2}{2\sigma^2}} + \frac{1-\rho}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x+\gamma)^2}{2\sigma^2}}$, then $\arg\max_{\rho} \{h(Y)\} = \frac{1}{2}$. \hfill $\Box$

The actual entropy $h(Y)$ of the random variable $Y$ distributed according to $f_Y$ in (10) for the optimal value $\rho = \frac{1}{2}$ is provided by the following fundamental proposition. The proof is omitted due to lack of space.

**Proposition 2** If the random variable $Y$ is distributed according to the density function $f_Y(x) = \frac{1}{2\sqrt{2\pi \sigma^2}} e^{-\frac{(x-\gamma)^2}{2\sigma^2}} + \frac{1}{2\sqrt{2\pi \sigma^2}} e^{-\frac{(x+\gamma)^2}{2\sigma^2}}$, then its entropy is equal to

$$h(Y) = \left(-\frac{2\gamma}{\sigma^2} e^{-\frac{2\gamma^2}{\sigma^2}} + \frac{2\rho}{\sigma^2} - 1\right) Q\left(\frac{\sqrt{\gamma}}{\sigma}\right) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)!} Q\left(\frac{\sqrt{\gamma}(2k+1)}{\sigma}\right) e^{\frac{2k\gamma(k+1)}{\sigma^2}} \log_2 e + \log_2 \left(2\sqrt{2\pi \sigma^2}\right). \hfill (11)$$

In (11), $Q(\alpha) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\frac{x^2}{2}} dx$. The infinite series included in (11) satisfies Abel’s uniform convergence test [19] over $(\rho, \sigma) \in [0, \infty) \times (0, \infty)$.

The channel capacity $C_B(p, \sigma)$ for binary transmissions over additive Gaussian noise subject to a constant-input-power constraint is obtained from Eqs. (7), (9) and Propositions 1, 2 as follows.

$$C_B(p, \sigma) \overset{\text{Prop. 1}}{=} \max_{f_X \in F_B(p)} \left\{ I(X; Y) \right\} = \log_2 (\sqrt{2\pi e\sigma^2})$$

$$C_B(p, \sigma) \overset{\text{Prop. 2}}{=} -\log_2 \left(\frac{2\gamma}{\sigma^2} e^{-\frac{2\gamma^2}{\sigma^2}} + \frac{2\rho}{\sigma^2} - 1\right) Q\left(\frac{\sqrt{\gamma}}{\sigma}\right) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)!} Q\left(\frac{\sqrt{\gamma}(2k+1)}{\sigma}\right) e^{\frac{2k\gamma(k+1)}{\sigma^2}} \log_2 e + 1. \hfill (12)$$

From (12) it can be shown that $C_B(p, \sigma)$ is a monotonically increasing function of $p$. We present this result in the form of a corollary below. The proof is omitted due to lack of space.

**Corollary 1** The channel capacity $C_B(p, \sigma)$ for binary transmissions over zero-mean additive Gaussian noise with variance $\sigma^2$ subject to a constant-input-power constraint $E\{X^2\} = p$ is a monotonically increasing function of $p$. \hfill $\Box$

Then, from (5)-(8) and Corollary 1 we obtain:

$$C_B(p, \sigma) \overset{(6)}{=} \max_{f_X \in F_B(p)} \left\{ I(X; Y) \right\} \overset{(5)}{=} \max_{\rho \in [0, 1]} \max_{f_X \in F_B(p)} \left\{ I(X; Y) \right\} \overset{(7)}{=} \max_{\rho \in [0, 1]} C_B^*(\rho, \sigma) \overset{(8)}{=} \max_{\rho \in [0, 1]} \max_{f_X \in F_B(p)} \left\{ I(X; Y) \right\} \overset{(9)}{=} \max_{\rho \in [0, 1]} C_B^*(\rho, \sigma). \hfill (13)$$

Hence, maximum-input-power or constant-input-power constraints result in the same channel capacity given by (12). Since $C_B(p, \sigma)$ (similarly to $C_B^*(p, \sigma)$) is a function of the input SNR constraint $\gamma = \frac{\alpha}{\sigma}$ we rewrite it as $C_B(\gamma)$ and express it in a closed form in the following lemma.

**Lemma 1** Consider two independent random variables $X \in \{\pm \sqrt{p}\}$ and $N \sim N(0, \sigma^2)$. If $\frac{\alpha^2}{\sigma^2} \leq \gamma$, then the channel with input $X$ and output $Y = X + N$ has capacity equal to

$$C_B(\gamma) = \left(-\frac{2\gamma}{\pi} e^{-\frac{2\gamma^2}{\pi}} + (2\gamma - 1) Q\left(\frac{\sqrt{\gamma}}{\sigma}\right) \right) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)!} Q\left(\sqrt{\gamma}(2k+1)\right) \log_2 e + 1. \hfill (14)$$

The channel capacity $C_B(\gamma)$ is achieved if and only if $X$ is uniformly distributed and $\frac{\alpha^2}{\sigma^2} = \gamma$. In addition, (i) $C_B(\gamma)$ is a monotonically increasing function of $\gamma \in [0, \infty)$, (ii) $0 = C_B(0) \leq C_B(\gamma) < 1$, $\gamma \in [0, \infty)$, (iii) $\lim C_B(\gamma) = 1$, and (iv) the infinite series included in $C_B(\gamma)$ is uniformly convergent on $[0, \infty)$.

For binary transmissions over additive Gaussian noise with a given maximum-input-SNR constraint $\gamma$, Eq. (14) determines the exact maximum achievable coding rate for reliable communication between the transmitter and the receiver. Of course, the expression for the channel capacity in Eq. (14) is equivalent to its original definition [1], [3], [6], [7], [14]

$$C_B(p, \sigma) = \int_{-\infty}^{\infty} \left(e^{-\frac{(x-\gamma)^2}{2\sigma^2}} - e^{-\frac{(x+\gamma)^2}{2\sigma^2}} + e^{-\frac{(x-\gamma)^2}{2\sigma^2}} - e^{-\frac{(x+\gamma)^2}{2\sigma^2}} \right) dx - \log_2 \sqrt{2\pi e\sigma^2} \hfill (15)$$

as well as its modified representation [12], [13]

$$C_B(\gamma) = \gamma - \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \log_2 \cosh \left(\gamma + \sqrt{\gamma} x\right) dx. \hfill (16)$$

However, in contrast to Eqs. (15) and (16) which still contain an integration, the new expression in Eq. (14) includes only the evaluation of the error function $Q(\cdot)$ at certain points and, in addition, the capacity can now be calculated with smaller complexity because -as shown in the next section- only the first six terms of the summation in Eq. (14) are practically required.

It should be emphasized that the channel capacity in Eq. (14) is achievable when soft-decision decoding is performed by the receiver. For binary transmissions over additive Gaussian noise, the channel capacity certainly decreases if hard-decision decoding is enforced. Indeed, the Gaussian channel with binary input and hard-decision decoding is equivalent to the binary symmetric channel with transition probability $Q(\sqrt{\gamma})$ [6], hence its capacity is given by [2]

$$C_H(\gamma) = 1 + Q(\sqrt{\gamma}) \log_2 Q(\sqrt{\gamma}) + (1 - Q(\sqrt{\gamma})) \log_2 (1 - Q(\sqrt{\gamma})). \hfill (17)$$

Comparison between $C_B(\gamma)$ in (14) and $C_H(\gamma)$ in (17) indicates the coding rate loss due to hard-decision decoding.
obtained directly from Eqs. (14) -originally derived in this channel capacity with real-valued input present work- and (17). As a reference, we include the Gaussian C

From another point of view, about the accuracy of such an approximation, an issue that is important to notice that the SNR gap between C

Let us keep only the first m terms of the summation in (14) to obtain a channel capacity approximation that we denote by

\[
\tilde{C}_B^{(m)}(\gamma) \triangleq \left(-\sqrt{\frac{2}{\pi}} e^{-\frac{\gamma}{4}} + (2\gamma - 1) Q(\sqrt{\gamma}) + \sum_{k=1}^{m} \frac{(-1)^k}{k(k+1)} Q(\sqrt{2}(2k+1)) e^{2\gamma(k+1)}\right) \log_2 e + 1
\]

where \( m = 1, 2, 3, \ldots \). Theoretical studies on \( \tilde{C}_B^{(m)}(\gamma) \) resulted in several interesting properties that we present in the form of a proposition below. The proof is omitted due to lack of space.

**Proposition 3** Let \( \tilde{C}_B^{(m)}(\gamma), m=1, 2, \ldots, \gamma > 0 \), be defined as in Eq. (18). Then, for any \( \gamma > 0 \)

(i) the sequence \( \tilde{C}_B^{(2m-1)}(\gamma), m=1, 2, \ldots \), is monotonically increasing,

(ii) the sequence \( \tilde{C}_B^{(2m)}(\gamma), m=1, 2, \ldots \), is monotonically decreasing,

(iii) the sequence \( \{\tilde{C}_B^{(m)}(\gamma) - C_B(\gamma)\}, m=1, 2, \ldots \), is monotonically decreasing, and

(iv) \( \lim_{m \to \infty} \tilde{C}_B^{(2m-1)}(\gamma) = \lim_{m \to \infty} \tilde{C}_B^{(2m)}(\gamma) = \lim_{m \to \infty} \tilde{C}_B^{(m)}(\gamma) = C_B(\gamma) \).

From Properties (i) and (iii), we conclude that \( \tilde{C}_B^{(2m-1)}(\gamma) < C_B(\gamma), m=1, 2, \ldots \). That is, truncation of the infinite series at an odd number \( 2m-1 \) of summation terms results in a lower bound on the channel capacity. According to Property (iv), the lower bound \( \tilde{C}_B^{(2m-1)}(\gamma) \) can be made arbitrarily tight by the selection of a sufficiently large number \( 2m-1 \). Similarly, Properties (ii) and (iii) imply that truncation at an even number \( 2m \) of terms results in an upper bound \( \tilde{C}_B^{(2m)}(\gamma) > C_B(\gamma), m=1, 2, \ldots \). Again, Property (iv) establishes that arbitrary tightness of the upper bound \( \tilde{C}_B^{(2m)}(\gamma) \) is achieved by the selection of an appropriately large number of terms \( 2m \).

We summarize the above findings in the following lemma.

**Lemma 2** Let \( \tilde{C}_B^{(m)}(\gamma), m=1, 2, \ldots, \gamma > 0 \), be defined as in Eq. (18). Then, for any \( \gamma > 0 \) the following statements hold true.

(i) \( 0 < \tilde{C}_B^{(1)}(\gamma) < \tilde{C}_B^{(3)}(\gamma) < \tilde{C}_B^{(5)}(\gamma) < \ldots < C_B(\gamma) < \ldots < \tilde{C}_B^{(6)}(\gamma) < \tilde{C}_B^{(4)}(\gamma) < \tilde{C}_B^{(2)}(\gamma) < 1 + \left(-\sqrt{\frac{2}{\pi}} e^{-\frac{\gamma}{4}} + (2\gamma - 1) Q(\sqrt{\gamma})\right) \log_2 e < 1 \).

(ii) For any \( \varepsilon > 0 \), there exists a positive integer \( m_0 \) such that for any \( m > m_0 \)

\[
C_B(\gamma) - \varepsilon < \tilde{C}_B^{(2m-1)}(\gamma) < C_B(\gamma) < \tilde{C}_B^{(2m)}(\gamma) < \tilde{C}_B^{(m)}(\gamma) < C_B(\gamma) + \varepsilon. \quad \square
\]

III. **Binary Transmissions Over Additive Gaussian Noise: Lower and Upper Bounds on the Channel Capacity**

Although the exact capacity \( C_B(\gamma) \) of the Gaussian channel with binary input subject to a maximum-input-SNR constraint \( \gamma \) was derived in Section II and presented in Lemma 1, computation limitations might impose restrictions on the number of calculable summation terms in (14). An unavoidable truncation of the infinite series in (14) results in an approximation of the exact capacity \( C_B(\gamma) \), raising questions about the accuracy of such an approximation, an issue that is addressed in this section.

In Fig. 1, Eq. (14) provides us with a useful upper bound on coding rates, especially for input SNR values within the range 0-10dB.

![Figure 1: Gaussian channel capacity with (i) real-valued input \( C_R(\gamma) \), (ii) binary input and soft-decision decoding \( C_B(\gamma) \), and (iii) binary input and hard-decision decoding \( C_H(\gamma) \) versus input SNR constraint \( \gamma \).](image-url)

To illustrate the tightness of the lower or upper bounds that we obtain by truncating the infinite series at an odd or even, respectively, number of terms, in Fig. 2 we plot the lower bounds \( \tilde{C}_B^{(m)}(\gamma) \), \( C_B(\gamma) \) and upper bounds \( \tilde{C}_B^{(2m)}(\gamma) \), \( \tilde{C}_B^{(m)}(\gamma) \) versus the maximum-input-SNR constraint \( \gamma \). As a reference, we include the exact channel capacity \( C_B(\gamma) \). As expected, when \( m = 5 \) or 6 the bounds become much tighter than when \( m = 1 \) or 2. In fact, the inclusion of the first \( m \geq 6 \) summation terms provides us with a lower or upper bound \( \tilde{C}_B^{(m)}(\gamma) \) indistinguishably away from the exact channel capacity \( C_B(\gamma) \) for the entire range of maximum-input-SNR constraints \( \gamma \).
Therefore, as long as at least six terms are included in the summation in (18), the truncated series results in an accurate evaluation of the exact channel capacity in (14).

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