New Bounds on the Total-Squared-Correlation and Optimum Design of DS-CDMA Binary Signature Sets

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ABSTRACT

The Welch lower bound on the total-squared-correlation (TSC) of signature sets is known to be tight for real-valued signatures and loose for binary signatures whose number is not a multiple of 4. In this paper, we derive new bounds on the TSC of binary signature sets for any number of signatures K and any signature length L. For almost all K, L in $\{1, 2, \ldots, 200\}$, we develop simple algorithms for the design of optimum binary signature sets that achieve the new bound.

Index Terms - Binary sequences, code division multi-access, codes, signal design.

I. INTRODUCTION

In direct-sequence code-division-multiple-access (DS-CDMA) systems, multiple users are assigned individual binary antipodal signatures (spreading codes) to access a common, in time and frequency, communication channel. In conjunction with channel and receiver design specifics, the overall system performance is determined by the selection of the user signature set. Since each user signal acts as interference for the signals of the other users, an appropriately selected/designed user signature set contains signatures with low pairwise cross-correlation.

A fundamental measure of the cross-correlation properties of a signature set is the *total-squared-correlation* (TSC).[1] If $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K\}$, $\mathbf{s}_i \in \mathbb{C}^L$, $\|\mathbf{s}_i\| = 1$, $i = 1, 2, \ldots, K$, is a set of K normalized (complex-valued in general) user signatures of length (processing gain) L, then the TSC of set $\mathcal{S}$ is the sum of the squared magnitudes of all inner products between signatures [2]:

$$\text{TSC}(\mathcal{S}) \triangleq \sum_{i=1}^{K}\sum_{j=1}^{K} \left|\mathbf{s}_i^H \mathbf{s}_j\right|^2 . \qquad (1)$$

In [3], Welch showed that

$$\text{TSC}(\mathcal{S}) \geq \frac{K^2}{L} \qquad (2)$$

and this lower bound was named [2] the “Welch bound” on the TSC of signature sets.

In [4], it is proven that if $K \geq L$, then there always exists a *real-valued* signature set that yields equality in the Welch bound.[2] Such optimum sets are called Welch-bound-equality (WBE) signature sets [2]. An iterative algorithm that converges -under appropriate initial conditions- to a real-valued WBE signature set is developed in [5]. WBE-optimum sets possess several important properties: *(i)* The cross-correlation metric TSC($\mathcal{S}$) is minimized and in this sense the multiple-access-interference (MAI) effects are minimized, *(ii)* the sum-capacity of the DS-CDMA channel is maximized [1], and *(iii)* the sum of transmitted energies required to surpass a pre-specified signal-to-interference-plus-noise-ratio (SINR) level at the receiver output is minimized [4].

While for real/complex-valued signature sets the Welch bound ($\frac{K^2}{L}$ for $K \geq L$ and K for $K < L$) is always achievable, this is not the case for binary antipodal signature sets. In [2], it is stated that if $K \geq L$ the Welch bound $\frac{K^2}{L}$ is achieved with binary antipodal signatures only if $K \equiv 0$ (mod 4) or $K = 1$ or 2. For the case of $K < L$, the lower bound K is achieved only if $L \equiv 0$ (mod 4) or $L = 1$ or 2. Optimum binary antipodal signature sets are constructed in [2] when the number of users is a power of 2 and equals or exceeds the system processing gain ($K = 2^n \geq L$, $n \leq L$).

In this work, we derive new bounds on the TSC of binary antipodal signature sets for all possible combinations of the values of K (number of users) and L (processing gain). In addition, for almost all $K, L \in \{1, 2, \ldots, 200\}$ we develop simple algorithms for the design of optimum binary antipodal signature sets that achieve the new bounds. Our algorithmic exceptions are: *(i)* $187 \leq K \leq 190$ and $K \geq L$, *(ii)* $187 \leq L \leq 190$ and $K \leq L$, *(iii)* $L = K \equiv 1$ (mod 4), *(iv)* $L = K \equiv 2$ (mod 4), *(v)* $L + 1 = K \equiv 2$ (mod 4), and *(vi)* $K + 1 = L \equiv 2$ (mod 4). The design procedure constitutes a constructive proof of the tightness of the new bounds for all $K, L \in \{1, 2, \ldots, 200\}$ that do not fall under exceptions *(i)*-*(vi)*.

The rest of this paper is organized as follows. In Section II, we derive the new TSC bounds for binary antipodal signature sets. In Section III, we present the algorithms for the design of TSC-optimum binary antipodal signature sets. Some concluding comments and examples are given in Section IV.

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[1] The term total-squared-correlation and the acronym TSC are due to [1].

[2] If $K < L$ the Welch bound $\frac{K^2}{L}$ becomes loose and a tighter bound exists: TSC($\mathcal{S}$) $\geq K$. In that case, the bound value K can be trivially achieved by any orthonormal set of K real/complex-valued signatures of length L.

II. NEW BOUNDS ON THE TSC OF BINARY ANTIPODAL SIGNATURE SETS

We consider a binary antipodal signature set $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_K\}$ with K normalized signatures $\mathbf{s}_i \in \left\{\pm\frac{1}{\sqrt{L}}\right\}^L$, $i = 1, 2, \dots, K$, where L is the CDMA system processing gain. Next, we derive new bounds on the TSC of the user signature set $\mathcal{S}$ for both "underloaded" ($K \leq L$) and "overloaded" ($K \geq L$) systems.

A. Underloaded system ($K \leq L$)

The total-squared-correlation of $\mathcal{S}$ is

$$\text{TSC}(\mathcal{S}) = \sum_{i=1}^{K} \sum_{j=1}^{K} \left(\mathbf{s}_i^T \mathbf{s}_j\right)^2 = K + \sum_{i=1}^{K} \sum_{j=1, j\neq i}^{K} \left(\mathbf{s}_i^T \mathbf{s}_j\right)^2. \quad (3)$$

The second, double-summation term is the total-squared-correlation between different users in $\mathcal{S}$. To obtain a bound on this term we need the following theorem. The proof is omitted due to lack of space.

Theorem 1: (On the Cross-Correlation Properties of Binary Antipodal Signature Sets)
Let $\mathcal{S} = \{\mathbf{s}_i\}_{i=1}^K$ be a binary antipodal signature set where $\mathbf{s}_i \in \{\pm A\}^L$, $i = 1, 2, \dots, K$, $K \leq L$, and $A \in \mathbb{C} - \{0\}$.
(i) If $\mathbf{s}_i^H \mathbf{s}_j \neq 0$, then

$$\left|\mathbf{s}_i^H \mathbf{s}_j\right| \geq \begin{cases} 2\,|A|^2, & L \equiv 0 \pmod 2 \\ \;\;|A|^2, & L \equiv 1 \pmod 2 \end{cases}, \quad 1 \leq i, j \leq K. \quad (4)$$

(ii) Consider the set C of all non-ordered pairs of signatures $\{\mathbf{s}_i, \mathbf{s}_j\}$, $i \neq j$, with non-zero cross-correlation:

$$C(\{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_K\}) \overset{\triangle}{=} \{\{\mathbf{s}_i, \mathbf{s}_j\} \colon i \neq j \text{ and } \mathbf{s}_i^H \mathbf{s}_j \neq 0\}. \quad (5)$$

If $|C|$ denotes the cardinality of set C, then

$$|C(\{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_K\})| \geq \begin{cases} 0, & L \equiv 0 \pmod 4 \\ \frac{K(K-2)}{4}, & L \equiv 2 \pmod 4 \\ & \text{and } K \equiv 0 \pmod 2 \\ \frac{(K-1)^2}{4}, & L \equiv 2 \pmod 4 \\ & \text{and } K \equiv 1 \pmod 2 \\ \frac{K(K-1)}{2}, & L \equiv 1 \pmod 2. \end{cases} \quad (6)$$

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We consider the even and odd cases for the processing gain L separately. If L is an even number ($L \equiv 0 \pmod 2$), by direct application of Theorem 1, Part (i) we obtain

$$\begin{aligned} \text{TSC}(\mathcal{S}) &\geq K + 2\,|C(\{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_K\})| \left(2\left|\frac{1}{\sqrt{L}}\right|^2\right)^2 \\ &= K + \frac{8}{L^2}\,|C(\{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_K\})|. \end{aligned} \quad (7)$$

Then, Theorem 1, Part (ii) gives

$$\text{TSC}(\mathcal{S}) \geq \begin{cases} K, & L \equiv 0 \pmod 4 \\ K + 2\frac{K(K-2)}{L^2}, & L \equiv 2 \pmod 4 \\ & \text{and } K \equiv 0 \pmod 2 \\ K + 2\left(\frac{K-1}{L}\right)^2, & L \equiv 2 \pmod 4 \\ & \text{and } K \equiv 1 \pmod 2. \end{cases} \quad (8)$$

TABLE I
UNDERLOADED DS-CDMA SYSTEM

Processing Gain	Number of Users	Lower Bound on TSC
$L \equiv 0 \pmod 4$	Any K	K
$L \equiv 2 \pmod 4$	$K \equiv 0 \pmod 2$	$K + 2\frac{K(K-2)}{L^2}$
	$K \equiv 1 \pmod 2$	$K + 2\left(\frac{K-1}{L}\right)^2$
$L \equiv 1 \pmod 2$	Any K	$K + \frac{K(K-1)}{L^2}$

If L is odd ($L \equiv 1 \pmod 2$), then from Theorem 1, Part (i) we obtain

$$\begin{aligned} \text{TSC}(\mathcal{S}) &\geq K + 2\,|C(\{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_K\})| \left(\left|\frac{1}{\sqrt{L}}\right|^2\right)^2 \\ &= K + \frac{2}{L^2}\,|C(\{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_K\})| \end{aligned} \quad (9)$$

and from Theorem 1, Part (ii)

$$\text{TSC}(\mathcal{S}) \geq K + \frac{K(K-1)}{L^2}, \quad L \equiv 1 \pmod 2. \quad (10)$$

Expressions (8) and (10) define the new bounds on the total-squared-correlation of binary antipodal signature sets for underloaded ($K \leq L$) CDMA systems. Notice that if the processing gain L is not a multiple of 4 (that is $L \not\equiv 0 \pmod 4$), the new bounds that we obtained here are tighter than the familiar bound TSC($\mathcal{S}$) $\geq K$ for real-valued signature sets.[3] Expressions (8) and (10) can also be seen as a proof that when the number of users is more than two and the signature length is not a multiple of 4, no orthogonal binary antipodal signature set exists. The new bounds of (8) and (10) for underloaded CDMA systems are summarized in Table I.

In Section III, we develop a simple procedure for the design of optimum binary antipodal signature sets that achieve the new bounds for all values of the number of users K and the processing gain L in $\{1, 2, \dots, 200\}$ for an underloaded system ($K \leq L$), except for K, L values that fall under cases *(ii), (iii), (iv),* and *(vi)* identified in the Introduction.

B. Overloaded system ($K \geq L$)

The total-squared-correlation of $\mathcal{S}$ can be written as

$$\begin{aligned} \text{TSC}(\mathcal{S}) &= \sum_{i=1}^{K}\sum_{j=1}^{K} \left(\mathbf{s}_i^T \mathbf{s}_j\right)^2 \quad (11) \\ &= \sum_{i=1}^{K}\sum_{j=1}^{K} \left(\sum_{l=1}^{L} \mathbf{s}_i(l)\mathbf{s}_j(l)\right)\left(\sum_{m=1}^{L} \mathbf{s}_i(m)\mathbf{s}_j(m)\right) \\ &= \sum_{l=1}^{L}\sum_{m=1}^{L} \left(\sum_{i=1}^{K} \mathbf{s}_i(l)\mathbf{s}_i(m)\right)\left(\sum_{j=1}^{K} \mathbf{s}_j(l)\mathbf{s}_j(m)\right). \end{aligned}$$

[3] Except for the trivial cases $L = 1$ and $L = 2$ where the bound is still TSC($\mathcal{S}$) $\geq K$ (and, in addition, is tight).

Let $\mathbf{D}$ denote the transpose of the signature matrix $[\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_K]$:

$$\mathbf{D} \triangleq [\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_K]^T . \tag{12}$$

The l-th column of $\mathbf{D}$ is $\mathbf{d}_l = [\mathbf{s}_1(l), \mathbf{s}_2(l), \dots, \mathbf{s}_K(l)]^T \in \left\{\pm\frac{1}{\sqrt{L}}\right\}^K$. Note that

$$\mathbf{d}_l(i) = \mathbf{s}_i(l) \in \left\{\pm\frac{1}{\sqrt{L}}\right\}, \quad i=1,\dots,K,\ l=1,\dots,L. \tag{13}$$

Then,

$$\begin{aligned}
\mathrm{TSC}(\mathcal{S}) &= \sum_{l=1}^{L}\sum_{m=1}^{L}\left(\sum_{i=1}^{K}\mathbf{d}_l(i)\mathbf{d}_m(i)\right)\left(\sum_{j=1}^{K}\mathbf{d}_l(j)\mathbf{d}_m(j)\right)\\
&= \sum_{l=1}^{L}\sum_{m=1}^{L}\left(\mathbf{d}_l^T\mathbf{d}_m\right)\left(\mathbf{d}_l^T\mathbf{d}_m\right)\\
&= \sum_{l=1}^{L}\left(\mathbf{d}_l^T\mathbf{d}_l\right)^2 + \sum_{l=1}^{L}\sum_{m=1,m\neq l}^{L}\left(\mathbf{d}_l^T\mathbf{d}_m\right)^2 \qquad (14)\\
&= \sum_{l=1}^{L}\left(\frac{K}{L}\right)^2 + \sum_{l=1}^{L}\sum_{m=1,m\neq l}^{L}\left(\mathbf{d}_l^T\mathbf{d}_m\right)^2\\
&= \frac{K^2}{L} + \sum_{l=1}^{L}\sum_{m=1,m\neq l}^{L}\left(\mathbf{d}_l^T\mathbf{d}_m\right)^2 .
\end{aligned}$$

We treat the K even and K odd cases separately. First, consider the K even case ($K \equiv 0 \pmod 2$). Recall that $L \leq K$ and $\mathbf{d}_l \in \left\{\pm\frac{1}{\sqrt{L}}\right\}^K$, $l=1,2,\dots,L$. By Theorem 1, Part (i) we obtain

$$\begin{aligned}
\mathrm{TSC}(\mathcal{S}) &\geq \frac{K^2}{L} + 2\left|C\left(\{\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_L\}\right)\right|\left(2\left|\frac{1}{\sqrt{L}}\right|^2\right)^2\\
&= \frac{K^2}{L} + \frac{8}{L^2}\left|C\left(\{\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_L\}\right)\right| . \qquad (15)
\end{aligned}$$

Then, by Theorem 1, Part (ii)

$$\mathrm{TSC}(\mathcal{S}) \geq \begin{cases} \frac{K^2}{L}, & K\equiv 0 \pmod 4)\\ \frac{K^2}{L} + 2\frac{L-2}{L}, & K\equiv 2 \pmod 4 \\ & \text{and } L\equiv 0 \pmod 2 \\ \frac{K^2}{L} + 2\left(\frac{L-1}{L}\right)^2, & K\equiv 2 \pmod 4 \\ & \text{and } L\equiv 1 \pmod 2). \end{cases} \tag{16}$$

Next, we consider the K odd case ($K \equiv 1 \pmod 2$). From Theorem 1, Part (i)

$$\begin{aligned}
\mathrm{TSC}(\mathcal{S}) &\geq \frac{K^2}{L} + 2\left|C\left(\{\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_L\}\right)\right|\left(\left|\frac{1}{\sqrt{L}}\right|^2\right)^2\\
&= \frac{K^2}{L} + \frac{2}{L^2}\left|C\left(\{\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_L\}\right)\right| \qquad (17)
\end{aligned}$$

and from Theorem 1, Part (ii) we conclude

$$\mathrm{TSC}(\mathcal{S}) \geq \frac{K^2}{L} + \frac{L-1}{L}, \quad K \equiv 1 \pmod 2. \tag{18}$$

TABLE II
OVERLOADED DS-CDMA SYSTEM

Number of Users	Processing Gain	Lower Bound on TSC
$K \equiv 0 \pmod 4$	Any L	$\frac{K^2}{L}$
$K \equiv 2 \pmod 4$	$L \equiv 0 \pmod 2$	$\frac{K^2}{L} + 2\frac{L-2}{L}$
	$L \equiv 1 \pmod 2$	$\frac{K^2}{L} + 2\left(\frac{L-1}{L}\right)^2$
$K \equiv 1 \pmod 2$	Any L	$\frac{K^2}{L} + \frac{L-1}{L}$

Expressions (16) and (18) define the new bounds on the total-squared-correlation of binary antipodal signature sets for overloaded ($K \geq L$) CDMA systems. When the number of users K is not a multiple of 4 ($K \not\equiv 0 \pmod 4$), the new bounds for binary antipodal signature sets that we obtained are tighter than the Welch bound TSC($\mathcal{S}$) $\geq \frac{K^2}{L}$ for real-valued signature sets.[4] The new bounds of (16) and (18) for overloaded CDMA systems are summarized in Table II.

As with the underloaded system case, in Section III we develop a procedure for the design of optimum binary antipodal signature sets that achieve the new bounds for all values of K and L in $\{1, 2, \dots, 200\}$ for overloaded systems ($K \geq L$), except for K, L values that fall under cases *(i)*, *(iii)*, *(iv)*, and *(v)* identified in the Introduction.

III. DESIGN OF MINIMUM-TSC BINARY ANTIPODAL SIGNATURE SETS

In this section we concentrate on the design of binary antipodal signature sets that achieve the new bounds derived in Section II (TSC-optimum binary signature sets). We begin with a definition and a proposition where we identify a sufficient condition under which the new bounds of Tables I and II become tight.

Definition 1: A Hadamard matrix of size N is an $N \times N$ matrix $\mathbf{A}$ with elements the real numbers $+1, -1$ and mutually orthogonal columns: $\mathbf{A}^T\mathbf{A} = N\mathbf{I}_{N\times N}$. □

Proposition 1: (Tightness Conditions for the TSC Bounds) Set $N \triangleq 4\left\lfloor\frac{\max\{K,L\}+1}{4}\right\rfloor$. If $N \geq \min\{K, L\}$ and there exists a Hadamard matrix of size N, then there exists a binary antipodal signature set $\mathcal{S} = \{\mathbf{s}_i\}_{i=1}^K$, $\mathbf{s}_i \in \left\{\pm\frac{1}{\sqrt{L}}\right\}^L$, $i = 1, 2, \dots, K$, that achieves equality in the corresponding TSC bound given by Table I or II. □

Below is a proof-by-construction of Proposition 1 that presents simple methods for the design of signature sets that achieve the lower bound on TSC under the conditions of the proposition. As in the previous section, the analysis and the developments are broken into two parts, one for underloaded ($K \leq L$) and one for overloaded

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[4] Except for the trivial cases $K = 1$ and $K = 2$ where the bound is still TSC($\mathcal{S}$) $\geq \frac{K^2}{L}$ (and, in addition, is tight).

($K \geq L$) systems. Each part is further itemized according to the exact relationship between $\max\{K, L\}$ and $N \triangleq 4\left\lfloor \frac{\max\{K,L\}+1}{4} \right\rfloor$.

A. Underloaded system ($K \leq L$)

For $K \leq L$, we have $N = 4\left\lfloor \frac{L+1}{4} \right\rfloor$ and $N \geq K$. By inspection, we observe that L takes one of the following four values: $L = N-1$, $L = N$, $L = N+1$, or $L = N+2$. Therefore, we need to design an optimum signature set (that is a set that achieves the corresponding bound on TSC) for all four of the above cases. We treat each case separately.

Let $\mathbf{H}_0$ be a Hadamard matrix of size N. Since $K \leq N$, we choose K *arbitrary columns* $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_K$ of $\mathbf{H}_0$, $\mathbf{h}_i \in \{\pm 1\}^N$, $i = 1, 2, \ldots, K$, and we define the $N \times K$ matrix $\mathbf{H} \triangleq [\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_K]$. Recall that $\mathbf{h}_i^T \mathbf{h}_j = 0$ for any $i = 1, 2, \ldots, K$, $j = 1, 2, \ldots, K$, and $i \neq j$.

1) $L = N-1$ Remove one *arbitrary row* from $\mathbf{H}$ to obtain the $L \times K$ matrix $\tilde{\mathbf{H}} = \left[\tilde{\mathbf{h}}_1, \tilde{\mathbf{h}}_2, \ldots, \tilde{\mathbf{h}}_K\right]$. Then, $\left|\tilde{\mathbf{h}}_i^T \tilde{\mathbf{h}}_j\right| = 1$ for any $i \neq j$. We design the $L \times K$ signature matrix as follows:

$$\boxed{[\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K] = \frac{1}{\sqrt{L}}\tilde{\mathbf{H}}.} \tag{19}$$

The signature set $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K\}$ consists of normalized signatures ($\|\mathbf{s}_i\| = 1, \quad i = 1, 2, \ldots, K$) and after straightforward calculations

$$\text{TSC}(\mathcal{S}) = K + \frac{K(K-1)}{L^2}. \tag{20}$$

Hence, TSC($\mathcal{S}$) is equal to the new bound in Table I for $L = N-1 \equiv 1$ (mod 2). We conclude that our binary signature set design in (19) is TSC-optimum.

2) $L = N$ We design[5] the $L \times K$ signature matrix directly from $\mathbf{H}$:

$$\boxed{[\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K] = \frac{1}{\sqrt{L}}\mathbf{H}.} \tag{21}$$

The signature set $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K\}$ consists of normalized signatures with zero cross-correlations, $\mathbf{s}_i^T \mathbf{s}_j = \frac{1}{L}\mathbf{h}_i^T \mathbf{h}_j = 0$, for any $i \neq j$. Therefore,

$$\text{TSC}(\mathcal{S}) = K \tag{22}$$

which is equal to the bound in Table I for $L = N \equiv 0$ (mod 4). The set in (21) is TSC-optimum.

3) $L = N+1$ Let $\mathbf{v} \in \{\pm 1\}^K$ be an *arbitrary* $K \times 1$ binary vector. We design the $L \times K$ signature matrix as follows:

$$\boxed{[\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K] = \frac{1}{\sqrt{L}}\begin{bmatrix} \mathbf{H} \\ \mathbf{v}^T \end{bmatrix}.} \tag{23}$$

Then, the signature set $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K\}$ consists of normalized signatures and after straightforward calculations we obtain

$$\text{TSC}(\mathcal{S}) = K + \frac{K(K-1)}{L^2}. \tag{24}$$

Hence, TSC($\mathcal{S}$) is equal to the bound in Table I for $L = N+1 \equiv 1$ (mod 2) and the optimality of the binary signature set design in (23) is established.

4a) $L = N+2$ and $K \equiv 0$ (mod 2) Let $\mathbf{v} \in \{\pm 1\}^{\frac{K}{2}}$ be an *arbitrary* $\frac{K}{2} \times 1$ binary vector. We design the $L \times K$ signature matrix as follows:

$$\boxed{[\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K] = \frac{1}{\sqrt{L}}\begin{bmatrix} \multicolumn{2}{c}{\mathbf{H}} \\ \mathbf{v}^T & \mathbf{v}^T \\ \mathbf{v}^T & -\mathbf{v}^T \end{bmatrix}.} \tag{25}$$

Then, the signature set $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K\}$ consists of normalized signatures and we can calculate

$$\text{TSC}(\mathcal{S}) = K + 2\frac{K(K-2)}{L^2}. \tag{26}$$

Hence, TSC($\mathcal{S}$) is equal to the bound in Table I for $L = N+2 \equiv 2$ (mod 4) and $K \equiv 0$ (mod 2). The set design in (25) is TSC-optimum.

4b) $L = N+2$ and $K \equiv 1$ (mod 2) Let $\mathbf{v} \in \{\pm 1\}^{\frac{K-1}{2}}$ be an *arbitrary* $\frac{K-1}{2} \times 1$ binary vector. We design the $L \times K$ signature matrix as follows:

$$\boxed{[\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K] = \frac{1}{\sqrt{L}}\begin{bmatrix} \multicolumn{3}{c}{\mathbf{H}} \\ \mathbf{v}^T & 1 & \mathbf{v}^T \\ \mathbf{v}^T & 1 & -\mathbf{v}^T \end{bmatrix}.} \tag{27}$$

The signature set $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K\}$ consists of normalized signatures and we calculate

$$\text{TSC}(\mathcal{S}) = K + 2\left(\frac{K-1}{L}\right)^2. \tag{28}$$

The TSC($\mathcal{S}$) is equal to the bound in Table I for $L = N+2 \equiv 2$ (mod 4) and $K \equiv 1$ (mod 2) and the set design in (27) is TSC-optimum.

B. Overloaded system ($K \geq L$)

For $K \geq L$, we have $N = 4\left\lfloor \frac{K+1}{4} \right\rfloor$ and $N \geq L$. By inspection, we observe that K takes one of the following four values: $K = N-1$, $K = N$, $K = N+1$, or $K = N+2$. Therefore, we need to design an optimum signature set for each one of the above four cases.

We start from an initial Hadamard matrix $\mathbf{H}_0$ of size N. Since $N \geq L$, we choose L *arbitrary columns* $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_L$ of $\mathbf{H}_0$, $\mathbf{h}_l \in \{\pm 1\}^N$, $l = 1, 2, \ldots, L$, and we define the $N \times L$ matrix $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_L]$. Recall that $\mathbf{h}_l^T \mathbf{h}_m = 0$ for any $l = 1, 2, \ldots, L$, $m = 1, 2, \ldots, L$, and $l \neq m$.

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[5] This case includes the familiar Rademacher-Walsh orthogonal codes [6], [7] for $L = 2^m$, $m = 2, 3, \ldots$, and $K \leq L$, used in current CDMA technology [8].

1) $K = N - 1$ Remove one *arbitrary row* from $\mathbf{H}$ to obtain the $K \times L$ matrix $\tilde{\mathbf{H}} = \left[\tilde{\mathbf{h}}_1, \tilde{\mathbf{h}}_2, \ldots, \tilde{\mathbf{h}}_L\right]$. Then, $\left|\tilde{\mathbf{h}}_l^T \tilde{\mathbf{h}}_m\right| = 1$ for any $l \neq m$. We design the $L \times K$ signature matrix as follows:

$$\boxed{[\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K] = \frac{1}{\sqrt{L}} \tilde{\mathbf{H}}^T.} \tag{29}$$

The signature set $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K\}$ has normalized signatures ($\|\mathbf{s}_i\| = 1, i = 1, 2, \ldots, K$) and

$$\text{TSC}(\mathcal{S}) = \frac{K^2}{L} + \frac{L-1}{L}. \tag{30}$$

Hence, TSC($\mathcal{S}$) is equal to the bound in Table II for $K = N - 1 \equiv 1 \pmod 2$. We conclude that the set that we designed in (29) is TSC-optimum.

2) $K = N$ We design the $L \times K$ signature matrix as follows:

$$\boxed{[\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K] = \frac{1}{\sqrt{L}} \mathbf{H}^T.} \tag{31}$$

The signature set $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K\}$ has normalized signatures. We calculate

$$\text{TSC}(\mathcal{S}) = \frac{K^2}{L} \tag{32}$$

which is equal to the bound in Table II for $K = N \equiv 0 \pmod 4$. The set-design in (31) is TSC-optimum.

3) $K = N + 1$ Let $\mathbf{v} \in \{\pm 1\}^L$ be an *arbitrary* $L \times 1$ binary vector. We design the $L \times K$ signature matrix as follows:

$$\boxed{[\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K] = \frac{1}{\sqrt{L}} \left[\mathbf{H}^T \quad \mathbf{v}\right].} \tag{33}$$

Then, the signature set $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K\}$ consists of normalized signatures and we can calculate

$$\text{TSC}(\mathcal{S}) = \frac{K^2}{L} + \frac{L-1}{L}. \tag{34}$$

Hence, TSC($\mathcal{S}$) is equal to the bound in Table II for $K = N + 1 \equiv 1 \pmod 2$ and the set in (33) is TSC-optimum.

4a) $K = N + 2$ and $L \equiv 0 \pmod 2$ Let $\mathbf{v} \in \{\pm 1\}^{\frac{L}{2}}$ be an *arbitrary* $\frac{L}{2} \times 1$ binary vector. We design the $L \times K$ signature matrix

$$\boxed{[\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K] = \frac{1}{\sqrt{L}} \begin{bmatrix} \mathbf{H}^T & \mathbf{v} & \mathbf{v} \\ & \mathbf{v} & -\mathbf{v} \end{bmatrix}.} \tag{35}$$

Then, the signature set $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K\}$ has normalized signatures and it can be shown that

$$\text{TSC}(\mathcal{S}) = \frac{K^2}{L} + 2\frac{L-2}{L}. \tag{36}$$

We conclude that TSC($\mathcal{S}$) is equal to the bound in Table II for $K = N + 2 \equiv 2 \pmod 4$ and $L \equiv 0 \pmod 2$. Therefore, the set-design in (35) is TSC-optimum.

4b) $K = N + 2$ and $L \equiv 1 \pmod 2$ Let $\mathbf{v} \in \{\pm 1\}^{\frac{L-1}{2}}$ be an *arbitrary* $\frac{L-1}{2} \times 1$ binary vector. We design the $L \times K$ signature matrix

$$\boxed{[\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K] = \frac{1}{\sqrt{L}} \begin{bmatrix} & \mathbf{v} & \mathbf{v} \\ \mathbf{H}^T & 1 & 1 \\ & \mathbf{v} & -\mathbf{v} \end{bmatrix}.} \tag{37}$$

The signature set $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K\}$ has normalized signatures and it can be shown that

$$\text{TSC}(\mathcal{S}) = \frac{K^2}{L} + 2\left(\frac{L-1}{L}\right)^2. \tag{38}$$

Hence, TSC($\mathcal{S}$) is equal to the bound in Table II for $K = N + 2 \equiv 2 \pmod 4$ and $L \equiv 1 \pmod 2$. Once again, the design in (37) is TSC-optimum.

IV. COMMENTS, CONCLUSIONS, AND EXAMPLES

In Section II we derived new bounds on the TSC of binary antipodal signature sets for both underloaded and overloaded CDMA systems (summarized in Table I and Table II, respectively). In Section III we identified sufficient conditions on the values of K (number of users) and L (processing gain) which guarantee that the corresponding new bounds on the TSC are tight. In addition, we were able to design optimum (minimum-TSC) binary antipodal signature sets for all values of K, L for which the sufficient conditions hold true.

The design of the optimum signature sets (and the tightness of the TSC bounds as it is described in Proposition 1) depends on the existence of a Hadamard matrix of size N. A necessary condition for a Hadamard matrix to exist is that its size is a multiple of 4 (except for the trivial cases of size 1 or 2). Indeed, N is a multiple of 4 by definition (see Proposition 1). Therefore, the necessary condition for the design algorithm to work is the existence of a Hadamard matrix for the specific value of N. Many Hadamard matrices are known for specific multiples of 4. Assuming that in CDMA applications values of K and L greater than 200 are of no much practical interest at present, we can mention that Hadamard matrices are known for all multiples of 4 less than or equal to 200, with the single exception of 188 [9]-[13]. We conclude that the only pairs of values of K and L in $\{1, 2, \ldots, 200\}$ for which we cannot guarantee that the new bounds are tight (or, alternatively, we do not have a design method for constructing optimum sets) are the ones covered by the following cases: *(i)* $187 \leq K \leq 190$ and $K \geq L$, *(ii)* $187 \leq L \leq 190$ and $K \leq L$, *(iii)* $L = K \equiv 1 \pmod 4$, *(iv)* $L = K \equiv 2 \pmod 4$, *(v)* $L + 1 = K \equiv 2 \pmod 4$, and *(vi)* $K + 1 = L \equiv 2 \pmod 4$.

It is interesting to note that these *(i)*-*(vi)* combinations constitute a small percentage (4.24%) among all possible combinations of K and L in $\{1, 2, \ldots, 200\}$. Therefore, the new bounds together with the sufficient conditions and the design procedures presented in this work provide us with useful tools that cover almost the whole range of possible CDMA setups of interest at present. An example of the optimum binary signature set for an underloaded CDMA system with processing gain $L = 18$ and $K = 11$ users is given by the 18×11 signature matrix $\mathbf{S}_1$ in Fig. If we increase the number of users to $K = 34$ the system becomes overloaded and the optimum signature set is given by the 18×34 signature matrix $\mathbf{S}_2$ in Fig. Both these optimum sets were obtained directly by the design procedure of Section III.

$$\mathbf{S}_1 = \begin{bmatrix}
+ & + & + & + & + & + & + & + & + & + & + \\
- & + & - & + & - & + & - & + & - & + & - \\
+ & - & - & + & + & - & - & + & + & - & - \\
- & - & + & + & - & - & + & + & - & - & + \\
+ & + & + & - & - & - & - & + & + & + & + \\
- & + & - & - & + & - & + & + & - & + & - \\
+ & - & - & - & - & + & + & + & + & - & - \\
- & - & + & - & + & + & - & + & - & - & + \\
+ & + & + & + & + & + & + & - & - & - & - \\
- & + & - & + & - & + & - & - & + & - & + \\
+ & - & - & + & + & - & - & - & - & + & + \\
- & - & + & + & - & - & + & - & + & + & - \\
+ & + & + & - & - & - & - & - & - & - & - \\
- & + & - & - & + & - & + & - & + & - & + \\
+ & - & - & - & - & + & + & - & - & + & + \\
- & - & + & - & + & + & - & - & + & + & - \\
+ & + & - & + & - & + & + & + & - & + & - \\
+ & + & - & + & - & + & - & - & + & - & +
\end{bmatrix}$$

Fig. 1. The optimum binary antipodal signature set for an underloaded system with processing gain $L = 18$ and $K = 11$ users.

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$$\mathbf{S}_2 = \begin{bmatrix}
+ & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & + \\
+ & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & - & - \\
+ & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + & + & + \\
+ & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - & - & - \\
+ & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + & + & + \\
+ & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + & + & + \\
+ & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - & - & - \\
+ & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - & - & - \\
+ & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + & + & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + & + & + \\
+ & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + & + & - \\
+ & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - & + & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - & - & + \\
+ & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + & + & - \\
+ & - & + & - & - & + & - & + & - & + & - & + & + & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + & + & - & + & - & - & + \\
+ & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - & + & - \\
+ & - & - & + & - & + & + & - & - & + & + & - & + & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - & + & - & - & + & + & - \\
+ & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & + \\
+ & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + \\
+ & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + & + & + & -
\end{bmatrix}$$

Fig. 2. The optimum binary antipodal signature set for an overloaded system with processing gain $L = 18$ and $K = 34$ users.