Rank-2-Optimal Adaptive Design of Binary Spreading Codes

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***Abstract*—Over the real/complex field, the spreading code that maximizes the signal-to-interference-plus-noise ratio (SINR) at the output of the maximum-SINR linear filter is the minimum-eigenvalue eigenvector of the interference autocovariance matrix. In the context of binary spreading codes, the maximization problem is NP-hard with complexity exponential in the code length. A new method for the optimization of binary spreading codes under a rank-2 approximation of the inverse interference autocovariance matrix is presented where the rank-2-optimal binary code is obtained in lower than quadratic complexity. Significant SINR performance improvement is demonstrated over the common binary hard-limited eigenvector design which is shown to be equivalent to the rank-1-optimal solution.**

***Index Terms*—Binary sequences, code-division multiple-access (CDMA), code-division multiplexing, signal waveform design, signal-to-interference-plus-noise ratio (SINR), signature sets, spread-spectrum communications.**

I. INTRODUCTION

THE performance of direct-sequence code-division-multiple-access (DS-CDMA) systems is determined by the set of user spreading codes in conjunction with channel and receiver design specifics. In recent literature, several methods have been presented for the design of real/complex-valued [1]–[14] or binary [8], [15]–[24] sets of DS-CDMA spreading codes under various optimization criteria. Among them, in [1]–[6], [14]–[22] the spreading code set is treated as a single matrix parameter to be optimized while the works in [7]–[12], [23], [24] present iterative algorithms that update the codes of the set individually in a round-robin fashion.

In particular, sets of spreading codes that minimize the total squared correlation (TSC) were designed in [1], [2], while distributed algorithms that iteratively decrease TSC by updating one-by-one the individual codes of the set were developed in [7]–[10]. Band-limited sets that maximize the user capacity of synchronous DS-CDMA systems were constructed in [3]. Optimal sets for asynchronous DS-CDMA systems were designed in [4]–[6], while the iterative method of [9], [10] was modified to suit asynchronous systems in [11] and multipath channels in [12]. Collaborative multibase designs were studied in [13]. In all the above developments, each user spreading code was allowed to take any value in the real vector space subject to a (unit-)norm constraint.

In [17], we concentrated on binary sets of spreading codes and derived new lower bounds on the TSC. Optimal binary sets that meet the TSC bound with equality were also designed in [17], as well as in [18], [19], while their sum capacity, total asymptotic efficiency, and maximum squared correlation were evaluated in [20]. A searching algorithm for minimum-TSC spreading code sets with low cross-correlation spectrum was presented in [21]. Interestingly, if we consider binary sets of spreading codes for transmission over a synchronous DS-CDMA channel with unequal channel gains for different users, then the minimum-TSC sets of [17]–[19] also minimize the corresponding total weighted squared correlation (TWSC) [14] as long as the system is underloaded and the spreading code length (processing gain) is odd or a multiple of four.

For overloaded systems, however, with unequal user gains, the minimum-TWSC optimality of minimum-TSC binary sets is lost; the design of optimal binary sets for such systems is still an open problem. In an attempt to design appropriate binary spreading codes for overloaded synchronous transmissions with unequal received power values as well as asynchronous transmissions over —potentially—multipath channels, suboptimal distributed algorithms were presented in [8], [23], [24]. In these works, the user codes are updated one-by-one, similar to the approach followed in [7]–[12] for real-valued spreading codes, simplifying the set design problem to the adaptive design of one spreading code in the presence of colored interference.

In this paper, we revisit the work of [8], [23], [24] and consider again the adaptive design of binary spreading codes that maximize the signal-to-interference-plus-noise ratio (SINR) at the output of the maximum-SINR (MSINR) filter. The optimal code is a function of the disturbance (interference-plus-noise) autocovariance matrix and its evaluation over the binary field is NP-hard [24]. Instead, in this present work we propose to eigendecompose and approximate the inverse disturbance autocovariance matrix by its *two* maximum-energy components alone. Then, we show how to optimize the binary spreading code under the *rank*-2 approximation of the inverse disturbance autocovariance matrix with lower than quadratic complexity. We demonstrate the significant SINR performance superiority of the proposed rank-2 adaptive design in comparison to direct hard-limiting of the maximum eigenvector of the inverse disturbance autocovariance matrix (or the minimum eigenvector of the interference autocovariance matrix) [8], [24], which we show to

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be equivalent to rank-1-optimal adaptive binary spreading code design. Moreover, simulation studies indicate that the obtained rank-2 spreading code can be the exact full-rank-optimal solution with significantly higher probability than the rank-1-optimal code (for example, when the rank-1-optimal code is globally optimal with probability 0.4, the returned rank-2 code is globally optimal with probability 0.85). In fact, simple iterative Hamming-distance-1 steepest descent search [22], [24] initialized at the proposed rank-2 spreading code raises the probability of convergence to the full-rank-optimal solution to as high as 0.95. Certainly, the proposed adaptive binary code design can serve as an appropriate initialization point for other, potentially more sophisticated, iterative binary search procedures.

The rest of the paper is organized as follows. Section II is devoted to signal model and problem formulation considerations. The proposed rank-2-optimal adaptive selection of the binary code is described in Section III. The performance of the proposed scheme is tested through simulations in Section IV. A few concluding remarks are drawn in Section V.

II. SIGNAL MODEL AND PROBLEM STATEMENT

Consider the vector signal model

$$\boldsymbol{r} = b\sqrt{P}\boldsymbol{s} + \boldsymbol{y} \tag{1}$$

where $b \in \{\pm 1\}$ is a uniformly distributed bit random variable, $\boldsymbol{s} \in \mathbb{R}^L$ a deterministic vector waveform (spreading code) with $\|\boldsymbol{s}\| = 1$ such that all collected energy is absorbed/represented by the scalar $P > 0$, and $\boldsymbol{y} \in \mathbb{R}^L$ is a zero-mean random disturbance vector with positive definite autocovariance matrix $\boldsymbol{R} = E\left\{\boldsymbol{y}\boldsymbol{y}^T\right\}$ ($E\{\cdot\}$ denotes statistical expectation and T is the transpose operator).

The general signal model of (1) covers, for example, synchronous DS-CDMA transmissions where a user of interest with spreading code $\boldsymbol{s}$ transmits over an additive noise channel in the presence of K interfering users. In that case, the overall additive disturbance term takes the form $\boldsymbol{y} = \sum_{k=1}^{K} b_k\sqrt{P_k}\boldsymbol{s}_k + \boldsymbol{n}$ where $b_k \in \{\pm 1\}$, $P_k > 0$, and $\boldsymbol{s}_k \in \mathbb{R}^L$ are the uniformly distributed user bit, received energy per bit, and normalized deterministic spreading code of the kth interferer, $k = 1, 2, \ldots, K$, and $\boldsymbol{n}$ represents additive zero-mean channel noise. Similarly, (1) can model asynchronous DS-CDMA transmissions when the receiver is synchronized with the signal of interest or can be extended to cover multipath transmissions if $\boldsymbol{s}$ is replaced by the effective (channel processed) user signature.

For an arbitrary spreading code $\boldsymbol{s}$, the linear filter/receiver $\boldsymbol{w}$ that exhibits maximum SINR at its output has the form

$$\boldsymbol{w}(\boldsymbol{s}) = c\boldsymbol{R}^{-1}\boldsymbol{s}, \quad c > 0, \tag{2}$$

and the (maximum) output SINR value is

$$\text{SINR}(\boldsymbol{s}) = \frac{E\left\{\left[\boldsymbol{w}^T(\boldsymbol{s})\, b\sqrt{P}\boldsymbol{s}\right]^2\right\}}{E\left\{\left[\boldsymbol{w}^T(\boldsymbol{s})\,\boldsymbol{y}\right]^2\right\}} = P\boldsymbol{s}^T\boldsymbol{R}^{-1}\boldsymbol{s}. \tag{3}$$

We recall that, if the additive disturbance vector $\boldsymbol{y}$ is Gaussian, then $\text{sgn}\left[\boldsymbol{w}^T(\boldsymbol{s})\,\boldsymbol{r}\right]$ is the minimum bit-error-rate (BER) detector with error probability given by $\text{BER}(\boldsymbol{s}) = Q\left(\sqrt{P\boldsymbol{s}^T\boldsymbol{R}^{-1}\boldsymbol{s}}\right)$ where $Q(a) = \int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$.

Our objective is to design the spreading code $\boldsymbol{s}$ so that the corresponding $\text{SINR}(\boldsymbol{s})$ value is maximized (and $\text{BER}(\boldsymbol{s})$ is minimized under the Gaussian additive disturbance assumption). Since the disturbance autocovariance matrix $\boldsymbol{R}$ is positive definite,

$$\boldsymbol{R} = \sum_{i=1}^{L} \lambda_i \boldsymbol{q}_i\boldsymbol{q}_i^T, \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_L > 0, \qquad \|\boldsymbol{q}_i\| = 1, \quad i = 1, 2, \ldots, L, \tag{4}$$

represents its eigendecomposition where λ_i and $\boldsymbol{q}_i$ are the ith eigenvalue and normalized eigenvector, respectively, of $\boldsymbol{R}$. Then, the *real-valued* spreading code $\boldsymbol{s}$ that maximizes $\text{SINR}(\boldsymbol{s})$ is given by [10]–[12], [24]

$$\boldsymbol{s}_{\mathbb{R}\text{-OPT}} \triangleq \arg \max_{\boldsymbol{s}\in\mathbb{R}^L, \|\boldsymbol{s}\|=1} \left\{P\boldsymbol{s}^T\boldsymbol{R}^{-1}\boldsymbol{s}\right\} = \boldsymbol{q}_L. \tag{5}$$

Therefore, the optimal real-valued code $\boldsymbol{s}_{\mathbb{R}\text{-OPT}}$ can be obtained with complexity that is dominated by the complexity of the eigendecomposition of the $L \times L$ matrix $\boldsymbol{R}$.

In this present work, we are interested in maximizing $\text{SINR}(\boldsymbol{s})$ subject to the requirement that $\boldsymbol{s}$ is *binary*. The problem of obtaining the optimal binary spreading code

$$\boldsymbol{s}_{\text{OPT}} \triangleq \arg \max_{\boldsymbol{s}\in\left\{\pm\frac{1}{\sqrt{L}}\right\}^L} \left\{P\boldsymbol{s}^T\boldsymbol{R}^{-1}\boldsymbol{s}\right\} \tag{6}$$

is NP-hard [23], [24] and can be solved through exhaustive search over all possible L-bit combinations. In the next section, we seek a computationally efficient alternative for the design of a spreading code $\boldsymbol{s} \in \left\{\pm\frac{1}{\sqrt{L}}\right\}^L$.

III. RANK-2-OPTIMAL DESIGN OF BINARY SPREADING CODES

Given the eigendecomposition of $\boldsymbol{R}$ in (4),

$$\boldsymbol{R}^{-1} = \sum_{i=1}^{L} \frac{1}{\lambda_i}\boldsymbol{q}_i\boldsymbol{q}_i^T, \qquad \frac{1}{\lambda_L} \geq \frac{1}{\lambda_{L-1}} \geq \cdots \geq \frac{1}{\lambda_1} > 0, \qquad \|\boldsymbol{q}_i\| = 1, \quad i = 1, 2, \ldots, L, \tag{7}$$

and the binary spreading code optimization problem in (6) can be rewritten as

$$\boldsymbol{s}_{\text{OPT}} = \arg \max_{\boldsymbol{s}\in\left\{\pm\frac{1}{\sqrt{L}}\right\}^L} \left\{\sum_{i=1}^{L} \frac{1}{\lambda_i}\left(\boldsymbol{s}^T\boldsymbol{q}_i\right)^2\right\} \tag{8}$$

where $0 \leq \left(\boldsymbol{s}^T\boldsymbol{q}_i\right)^2 \leq 1$, $i = 1, 2, \ldots, L$. Therefore, the optimal binary vector $\boldsymbol{s}_{\text{OPT}}$ maximizes the sum of its weighted projections on the eigenvectors $\boldsymbol{q}_i$ with weights $1/\lambda_i$, $i = 1, \ldots, L$.

If we simplify the optimization problem by keeping only the strongest term $\frac{1}{\lambda_L}\left(\boldsymbol{s}^T\boldsymbol{q}_L\right)^2$ in (8) (which is equivalent to using the approximation $\boldsymbol{R}^{-1} \simeq \frac{1}{\lambda_L}\boldsymbol{q}_L\boldsymbol{q}_L^T$ of (7) in (6)), we obtain the *rank-1-optimal* spreading code

$$\boldsymbol{s}_1 \triangleq \arg \max_{\boldsymbol{s}\in\left\{\pm\frac{1}{\sqrt{L}}\right\}^L} \left\{\left(\boldsymbol{s}^T\boldsymbol{q}_L\right)^2\right\} = \pm\frac{1}{\sqrt{L}}\text{sgn}(\boldsymbol{q}_L) \tag{9}$$

where $\operatorname{sgn}(\cdot)$ denotes the sign operator. Hence, we showed that the rank-1-optimal binary code is simply the minimum-eigenvalue eigenvector of $\boldsymbol{R}$ passed through a sign hard-limiter. From another point of view, if we follow the common approach of: i) first relaxing the binary constraint by allowing $\boldsymbol{s} \in \mathbb{R}^L$, $\|\boldsymbol{s}\| = 1$, ii) then solving the relaxed problem to obtain $\boldsymbol{s}_{\mathbb{R}-\mathrm{OPT}} = \boldsymbol{q}_L$, and iii) finally mapping (quantizing) $\boldsymbol{s}_{\mathbb{R}-\mathrm{OPT}} = \boldsymbol{q}_L$ to the binary field by taking the sign of its components, then we again obtain the rank-1-optimal binary spreading code $\frac{1}{\sqrt{L}}\operatorname{sgn}(\boldsymbol{s}_{\mathbb{R}-\mathrm{OPT}}) = \frac{1}{\sqrt{L}}\operatorname{sgn}(\boldsymbol{q}_L) = \boldsymbol{s}_1$. Therefore, the relaxation approach in [8], [24] is equivalent to rank-1 approximation of $\boldsymbol{R}^{-1}$.

To obtain a binary spreading code with higher SINR than $\mathrm{SINR}(\boldsymbol{s}_1)$, in this present work we propose to keep the two strongest terms $\frac{1}{\lambda_L}\left(\boldsymbol{s}^T\boldsymbol{q}_L\right)^2 + \frac{1}{\lambda_{L-1}}\left(\boldsymbol{s}^T\boldsymbol{q}_{L-1}\right)^2$ in (8) or—equivalently—use rank-2 approximation of $\boldsymbol{R}^{-1}$ in (6) $\boldsymbol{R}^{-1} \simeq \frac{1}{\lambda_L}\boldsymbol{q}_L\boldsymbol{q}_L^T + \frac{1}{\lambda_{L-1}}\boldsymbol{q}_{L-1}\boldsymbol{q}_{L-1}^T$. Below, we show that the rank-2-optimal binary vector can always be obtained with less than quadratic complexity (less than the complexity required for the eigendecomposition of $\boldsymbol{R}$).

We begin by defining the complex vector

$$\boldsymbol{z} \triangleq \frac{1}{\sqrt{\lambda_L}}\boldsymbol{q}_L + j\frac{1}{\sqrt{\lambda_{L-1}}}\boldsymbol{q}_{L-1}. \tag{10}$$

Then, the problem of interest is converted to

$$\begin{aligned} &\arg\max_{\boldsymbol{s}\in\left\{\pm\frac{1}{\sqrt{L}}\right\}^L}\left\{\frac{1}{\lambda_L}\left(\boldsymbol{s}^T\boldsymbol{q}_L\right)^2 + \frac{1}{\lambda_{L-1}}\left(\boldsymbol{s}^T\boldsymbol{q}_{L-1}\right)^2\right\} \\ &\quad= \arg\max_{\boldsymbol{s}\in\left\{\pm\frac{1}{\sqrt{L}}\right\}^L}\left\{\left|\boldsymbol{s}^T\boldsymbol{z}\right|^2\right\} \\ &\quad= \arg\max_{\boldsymbol{s}\in\left\{\pm\frac{1}{\sqrt{L}}\right\}^L}\left\{\left|\boldsymbol{s}^T\boldsymbol{z}\right|\right\}. \end{aligned} \tag{11}$$

In an effort to solve (11) in less than quadratic complexity, we consider the auxiliary-variable technique that has been used in a communications theory context before in [25] and then again in [26]. Let $\boldsymbol{s} = [s_1 \ \ s_2 \ \ \ldots \ \ s_L]^T$ and $\boldsymbol{z} = [z_1 \ \ z_2 \ \ \ldots \ \ z_L]^T$ where

$$z_i = |z_i|\,e^{j\phi_i}, \quad -\frac{\pi}{2} \le \phi_i < \frac{3\pi}{2}, \qquad i = 1,2,\ldots,L. \tag{12}$$

We introduce an auxiliary variable $\phi \in [-\pi,\pi)$ and rewrite the quantity to be maximized as

$$\begin{aligned} \left|\boldsymbol{s}^T\boldsymbol{z}\right| &= \max_{\phi\in[-\pi,\pi)}\left\{\operatorname{Re}\left\{\boldsymbol{s}^T\boldsymbol{z}e^{-j\phi}\right\}\right\} \\ &= \max_{\phi\in[-\pi,\pi)}\left\{\sum_{i=1}^{L} s_i\,|z_i|\cos(\phi-\phi_i)\right\} \end{aligned} \tag{13}$$

where $\operatorname{Re}\{\cdot\}$ extracts the real part of a complex number. Then

$$\begin{aligned} &\max_{\boldsymbol{s}\in\left\{\pm\frac{1}{\sqrt{L}}\right\}^L}\left\{\left|\boldsymbol{s}^T\boldsymbol{z}\right|\right\} \\ &\quad= \max_{\boldsymbol{s}\in\left\{\pm\frac{1}{\sqrt{L}}\right\}^L}\ \max_{\phi\in[-\pi,\pi)}\left\{\sum_{i=1}^{L} s_i\,|z_i|\cos(\phi-\phi_i)\right\} \\ &\quad= \max_{\phi\in[-\pi,\pi)}\left\{\sum_{i=1}^{L}|z_i|\max_{s_i\in\left\{\pm\frac{1}{\sqrt{L}}\right\}}\left\{s_i\cos(\phi-\phi_i)\right\}\right\} \qquad (14) \\ &\quad= \max_{\phi\in[-\pi,\pi)}\left\{\sum_{i=1}^{L}|z_i|\,\frac{1}{\sqrt{L}}\left|\cos(\phi-\phi_i)\right|\right\} \qquad (15) \end{aligned}$$

since for any $\phi \in [-\pi,\pi)$ the optimal s_i value in (14) is $s_i(\phi) = \frac{1}{\sqrt{L}}\operatorname{sgn}(\cos(\phi-\phi_i))$, $i = 1,2,\ldots,L$. The final quantity $\sum_{i=1}^{L}|z_i|\frac{1}{\sqrt{L}}\left|\cos(\phi-\phi_i)\right|$ in (15) is maximized for a particular value $\phi_{\mathrm{OPT}} \in [-\pi,\pi)$ and $\boldsymbol{s}(\phi_{\mathrm{OPT}}) = [s_1(\phi_{\mathrm{OPT}}) \ \ s_2(\phi_{\mathrm{OPT}}) \ \ \ldots \ \ s_L(\phi_{\mathrm{OPT}})]^T$ is the *rank-2-optimal binary vector* we are searching for in (11). We will now show that we can always construct a set of L spreading codes $\mathcal{U} = \{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_L\}$, $\boldsymbol{u}_i \in \left\{\pm\frac{1}{\sqrt{L}}\right\}^L$, and guarantee that $\boldsymbol{s}(\phi_{\mathrm{OPT}}) \in \mathcal{U}$. Therefore, the maximization in (11) can be carried out over a set of L candidates only without loss of optimality.

Partition $\mathbb{Z}_L = \{1,2,\ldots,L\}$ into

$$I_1 = \left\{i : \phi_i \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right)\right\} \text{ and}$$

$$I_2 = \left\{i : \phi_i \in \left[\frac{\pi}{2},\frac{3\pi}{2}\right)\right\} = \mathbb{Z}_L - I_1 \tag{16}$$

and define the set of angles

$$\hat{\phi}_i \triangleq \begin{cases}\phi_i, & i \in I_1,\\ \phi_i - \pi, & i \in I_2,\end{cases} \qquad i = 1,2,\ldots,L \tag{17}$$

such that $-\frac{\pi}{2} \le \hat{\phi}_i < \frac{\pi}{2}$, $i = 1,2,\ldots,L$. Define, for notational simplicity, the vector operation $\hat{\boldsymbol{s}}(\phi) \triangleq [\hat{s}_1(\phi)\,\hat{s}_2(\phi)\ \ldots\ \hat{s}_L(\phi)]^T$ with $\hat{s}_i(\phi) \triangleq \operatorname{sgn}\left(\cos\left(\phi - \hat{\phi}_i\right)\right)$, $i = 1,2,\ldots,L$, $\phi \in [-\pi,\pi)$. Then

$$s_i(\phi) = \begin{cases}\frac{\hat{s}_i(\phi)}{\sqrt{L}}, & i \in I_1,\\ -\frac{\hat{s}_i(\phi)}{\sqrt{L}} & i \in I_2,\end{cases} \qquad i = 1,2,\ldots,L. \tag{18}$$

Consider a mapping e from $\mathbb{Z}_L$ to $\mathbb{Z}_L$ that sorts $\hat{\phi}_1,\hat{\phi}_2,\ldots,\hat{\phi}_L$: $-\frac{\pi}{2} \le \hat{\phi}_{e(1)} \le \hat{\phi}_{e(2)} \le \cdots \le \hat{\phi}_{e(L)} < \frac{\pi}{2}$. In (14), (15), maximization with respect to ϕ can be carried out over any subinterval of length π. We choose $\hat{\phi}_{e(1)} - \frac{\pi}{2}$ and $\hat{\phi}_{e(1)} + \frac{\pi}{2}$ as the limits of such a subinterval and rewrite the original optimization problem of (14) as

$$\begin{aligned} &\max_{\boldsymbol{s}\in\left\{\pm\frac{1}{\sqrt{L}}\right\}^L}\left\{\left|\boldsymbol{s}^T\boldsymbol{z}\right|\right\} \\ &= \max_{\phi\in\left[\hat{\phi}_{e(1)}-\frac{\pi}{2},\hat{\phi}_{e(1)}+\frac{\pi}{2}\right)}\left\{\sum_{i=1}^{L}|z_i|\max_{s_i\in\left\{\pm\frac{1}{\sqrt{L}}\right\}}\left\{s_i\cos(\phi-\phi_i)\right\}\right\}. \end{aligned} \tag{19}$$

By examining the subintervals $\left[\hat{\phi}_{e(1)} - \frac{\pi}{2}, \hat{\phi}_{e(2)} - \frac{\pi}{2}\right)$, $\left[\hat{\phi}_{e(2)} - \frac{\pi}{2}, \hat{\phi}_{e(3)} - \frac{\pi}{2}\right)$, $\ldots$, $\left[\hat{\phi}_{e(L-1)} - \frac{\pi}{2}, \hat{\phi}_{e(L)} - \frac{\pi}{2}\right)$, $\left[\hat{\phi}_{e(L)} - \frac{\pi}{2}, \hat{\phi}_{e(1)} + \frac{\pi}{2}\right)$, we get (20) at the top of the following page. We collect the L binary vectors that appear in the L cases in (20),

$$\tilde{\boldsymbol{u}}_i \triangleq [\underbrace{+1\cdots+1}_{i}\underbrace{-1\ldots-1}_{L-i}]^T, \qquad i = 1,2,\ldots,L$$

and create the matrix

$$\tilde{\boldsymbol{U}} \triangleq [\tilde{\boldsymbol{u}}_1 \ \ \tilde{\boldsymbol{u}}_2 \ \ \ldots \ \ \tilde{\boldsymbol{u}}_L] \tag{21}$$

$$\left[\hat{s}_{e(1)}(\phi) \quad \hat{s}_{e(2)}(\phi) \quad \dots \quad \hat{s}_{e(L)}(\phi)\right] = \begin{cases} [+1 \;\; -1 \;\; -1 \;\; \cdots \;\; -1 \;\; -1], & \hat{\phi}_{e(1)} - \frac{\pi}{2} \le \phi < \hat{\phi}_{e(2)} - \frac{\pi}{2} \\ [+1 \;\; +1 \;\; -1 \;\; \cdots \;\; -1 \;\; -1], & \hat{\phi}_{e(2)} - \frac{\pi}{2} \le \phi < \hat{\phi}_{e(3)} - \frac{\pi}{2} \\ \qquad\qquad \vdots & \\ [+1 \;\; +1 \;\; +1 \;\; \cdots \;\; +1 \;\; -1], & \hat{\phi}_{e(L-1)} - \frac{\pi}{2} \le \phi < \hat{\phi}_{e(L)} - \frac{\pi}{2} \\ [+1 \;\; +1 \;\; +1 \;\; \cdots \;\; +1 \;\; +1], & \hat{\phi}_{e(L)} - \frac{\pi}{2} \le \phi < \hat{\phi}_{e(1)} + \frac{\pi}{2}. \end{cases} \tag{20}$$

whose ith row we denote by $\tilde{\boldsymbol{d}}_i^T$, $i = 1, 2, \dots, L$. Then, we reorganize $\tilde{\boldsymbol{U}}$ to

$$\hat{\boldsymbol{U}} = [\hat{\boldsymbol{u}}_1 \quad \hat{\boldsymbol{u}}_2 \quad \dots \quad \hat{\boldsymbol{u}}_L] \triangleq = \left[\hat{\boldsymbol{d}}_1 \quad \hat{\boldsymbol{d}}_2 \quad \dots \quad \hat{\boldsymbol{d}}_L\right]^T$$

by defining the binary vectors

$$\hat{\boldsymbol{d}}_i \triangleq \tilde{\boldsymbol{d}}_{e^{-1}(i)}, \qquad i = 1, 2, \dots, L \tag{22}$$

where $e^{-1} : \mathbb{Z}_L \to \mathbb{Z}_L$ is the inverse sorting mapping (note that $\hat{\boldsymbol{s}}(\phi) \in \{\hat{\boldsymbol{u}}_1, \hat{\boldsymbol{u}}_2, \dots, \hat{\boldsymbol{u}}_L\}$ for any $\phi \in \left[\hat{\phi}_{e(1)} - \frac{\pi}{2}, \hat{\phi}_{e(1)} + \frac{\pi}{2}\right)$). Finally, we define

$$\boldsymbol{d}_i \triangleq \begin{cases} \frac{1}{\sqrt{L}}\hat{\boldsymbol{d}}_i, & i \in I_1, \\ -\frac{1}{\sqrt{L}}\hat{\boldsymbol{d}}_i, & i \in I_2, \end{cases} \quad i = 1, 2, \dots, L \tag{23}$$

and construct

$$\boldsymbol{U} = [\boldsymbol{u}_1 \quad \boldsymbol{u}_2 \quad \dots \quad \boldsymbol{u}_L] \triangleq [\boldsymbol{d}_1 \quad \boldsymbol{d}_2 \quad \dots \quad \boldsymbol{d}_L]^T. \tag{24}$$

The set $\mathcal{U} \triangleq \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_L\}$ contains $\boldsymbol{s}$ for any $\phi \in \left[\hat{\phi}_{e(1)} - \frac{\pi}{2}, \hat{\phi}_{e(1)} + \frac{\pi}{2}\right)$. But $\phi_{\text{OPT}} \in \left[\hat{\phi}_{e(1)} - \frac{\pi}{2}, \hat{\phi}_{e(1)} + \frac{\pi}{2}\right)$, which implies that the rank-2-optimal code $\boldsymbol{s}(\phi_{\text{OPT}}) \in \mathcal{U}$. Hence, (11) becomes

$$\arg \max_{\boldsymbol{s} \in \left\{\pm\frac{1}{\sqrt{L}}\right\}^L} \left\{\left|\boldsymbol{s}^T\boldsymbol{z}\right|\right\} = \arg\max_{\boldsymbol{s} \in \mathcal{U}} \left\{\left|\boldsymbol{s}^T\boldsymbol{z}\right|\right\}. \tag{25}$$

We conclude that the maximization task in (11) has been converted to an equivalent linear-complexity maximization problem in (25). The complexity of the construction of $\mathcal{U}$ is dominated by the complexity of the sorting function e which is of order $O(L\log_2 L)$. Therefore, we have described a method to obtain the rank-2-optimal binary vector with complexity that is dominated by the complexity of the eigendecomposition of the $L \times L$ matrix $\boldsymbol{R}$ alone. Given that the rank-2-optimal code is in $\mathcal{U}$, the proposal

$$\boldsymbol{s}_2 \triangleq \arg\max_{\boldsymbol{s} \in \mathcal{U}} \boldsymbol{s}^T\boldsymbol{R}^{-1}\boldsymbol{s} \tag{26}$$

returns the rank-2-optimal solution or better, in the sense that with nonzero probability $\mathcal{U}$ may contain a code with higher output SINR than the rank-2-optimal code.

In summary, the sequence of the proposed calculations is as follows. The autocovariance matrix $\boldsymbol{R}$ is eigendecomposed as in (4) followed by the computation of the $\boldsymbol{z}$ vector by (10). The sets I_1 and I_2 are constructed through (16); the angles $\hat{\phi}_1, \hat{\phi}_2, \dots, \hat{\phi}_L$ are calculated by (17) and sorted to $\hat{\phi}_{e(1)} \le \hat{\phi}_{e(2)} \le \cdots \le \hat{\phi}_{e(L)}$. Next, $\tilde{\boldsymbol{U}}$ is constructed as in (21) and reorganized to $\hat{\boldsymbol{U}}$ via (22); then, $\boldsymbol{U}$ is found by (24). The columns of $\boldsymbol{U}$, $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_L$, are used to calculate the quantities $\boldsymbol{u}_1^T\boldsymbol{R}^{-1}\boldsymbol{u}_1, \boldsymbol{u}_2^T\boldsymbol{R}^{-1}\boldsymbol{u}_2, \dots, \boldsymbol{u}_L^T\boldsymbol{R}^{-1}\boldsymbol{u}_L$ that are compared to each other to identify the maximum value among them, say $\boldsymbol{u}_j^T\boldsymbol{R}^{-1}\boldsymbol{u}_j$ for some $j \in \{1, 2, \dots, L\}$. The latter determines the rank-2-optimal-or-better binary spreading code $\boldsymbol{s}_2 = \boldsymbol{u}_j$.

The described rank-2 single-user spreading code design method may serve as the basis for a multiuser spreading code set adaptation scheme. In such an approach, each user's spreading code is updated one after the other. Since each spreading code update results in changes to the interference-plus-noise statistics seen by the other users, a new update cycle may follow. Convergence analysis of such an iterative multiuser adaptation procedure seems to be a worthy endeavor (pertinent studies for multiuser signature adaptation over the real field can be found in [27], [28], [10]).

IV. SIMULATION STUDIES

We consider a DS-CDMA system where the user of interest transmits over an additive zero-mean white Gaussian noise channel with variance σ^2 in the presence of K synchronous interferers. The system processing gain (spreading code length) is $L = 16$. The received signal-to-noise ratio (SNR) of the user of interest, $\text{SNR} \triangleq \frac{P}{\sigma^2}$, is set to 10 dB, while the received SNRs of the K interferers, $\text{SNR}_k \triangleq \frac{P_k}{\sigma^2}$, $k = 1, 2, \dots, K$, are uniformly spaced between 8 dB and 11 dB. The interfering spreading codes are randomly generated.

In our studies, we compare the SINR performance of: i) the *optimal* binary spreading code $\boldsymbol{s}_{\text{OPT}}$ of (6) obtained through exhaustive search over all possible L-bit combinations; ii) the *rank*-1-*optimal* binary spreading code $\boldsymbol{s}_1$ of (9) obtained by applying the sign operator on the minimum-eigenvalue eigenvector of the interference-plus-noise autocovariance matrix [8], [24]; and iii) the proposed *rank*-2 binary spreading code $\boldsymbol{s}_2$ of (26) obtained by the procedure developed in Section III. For comparison purposes, we evaluate the SINR loss, $\text{SINR}(\boldsymbol{s}_{\mathbb{R}\text{-OPT}}) - \text{SINR}(\boldsymbol{s})$, of $\boldsymbol{s} = \boldsymbol{s}_{\text{OPT}}$, $\boldsymbol{s}_1$, and $\boldsymbol{s}_2$ with respect to the SINR of the optimal real spreading code $\boldsymbol{s}_{\mathbb{R}\text{-OPT}}$ in (5). The results that we present are averages over 1000 randomly generated interference signature-set realizations.

In Fig. 1, we plot the SINR loss of the rank-1, rank-2, and full-rank-optimal binary spreading codes as a function of the number of interferers K. Since we are particularly interested in overloaded systems, we vary K from 16 to 40 interferers. We observe that the proposed rank-2 binary spreading code exhibits practically the same loss as the full-rank-optimal binary solution and together are within about 1.1 dB from the real-valued optimal vector.

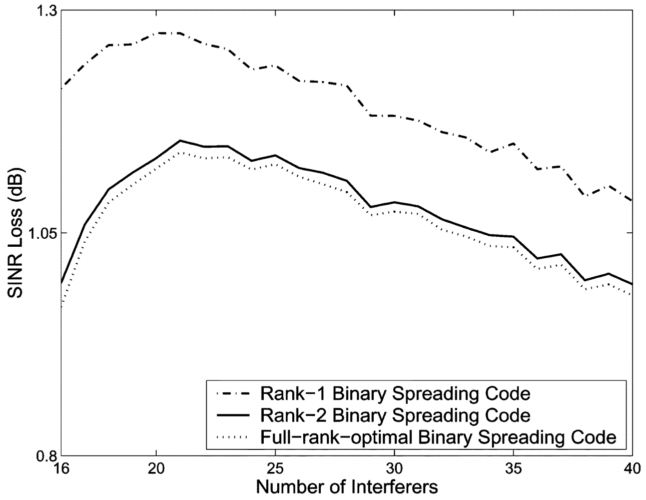

Fig. 1. SINR loss of rank-1, rank-2, and full-rank binary spreading code designs versus number of interferers (signature length $L = 16$).

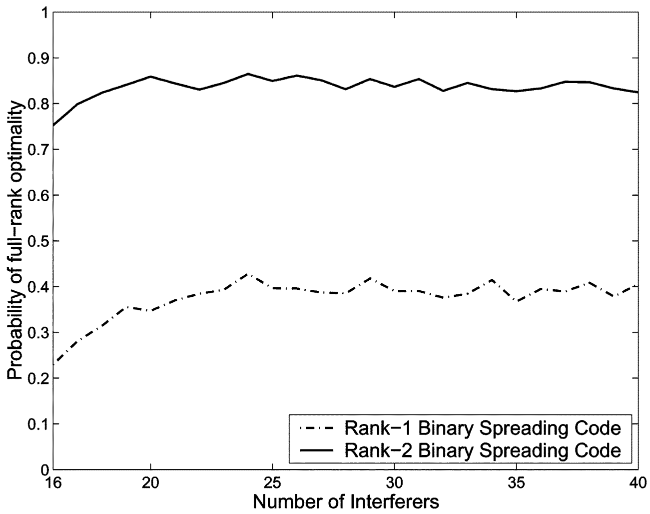

Fig. 2. Probability of full-rank optimality of rank-1 and rank-2 binary spreading code designs versus number of interferers.

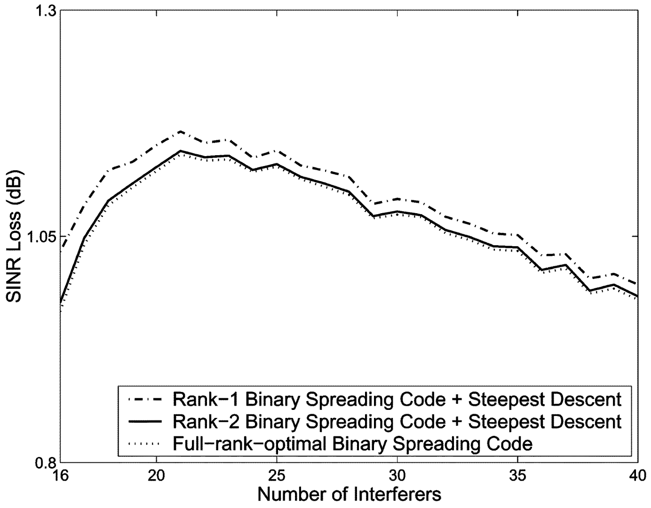

Fig. 3. SINR loss of steepest descent upon convergence with rank-1 and rank-2 binary code initialization.

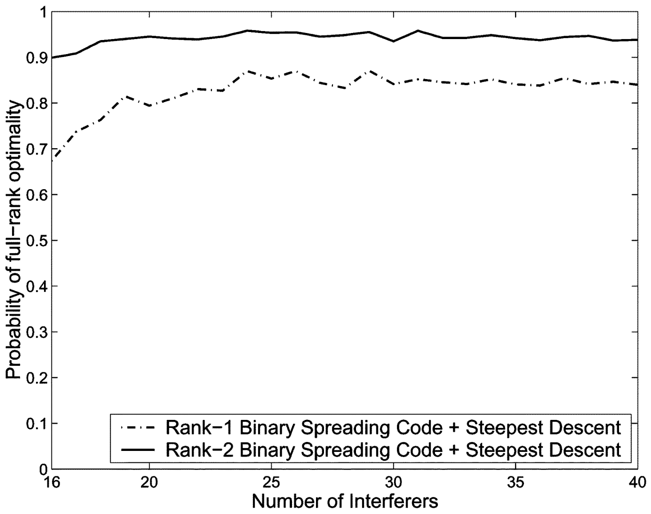

Fig. 4. Probability of full-rank optimality of steepest descent search upon convergence with rank-1 and rank-2 binary code initialization.

In Fig. 2, we plot the probability of global, full-rank, optimality $\Pr\{\boldsymbol{s} = \boldsymbol{s}_{\mathrm{OPT}}\}$ for the rank-1 and rank-2 binary spreading codes as a function of the number of interferers K. We observe that $0.23 \leq \Pr\{\boldsymbol{s}_1 = \boldsymbol{s}_{\mathrm{OPT}}\} \leq 0.43$, while $0.75 \leq \Pr\{\boldsymbol{s}_2 = \boldsymbol{s}_{\mathrm{OPT}}\} \leq 0.86$ as K is varied from 16 to 40 interferers. Therefore, with the proposed optimization of the binary spreading code under the rank-2 approximation of the inverse disturbance autocovariance matrix, we have practically doubled the probability that the designed spreading code is full-rank optimal with only $O(L \log_2 L)$ additional computational cost.

To the extend that neither the rank-1 nor the rank-2 binary spreading code design is globally optimal with probability one, the returned codes can potentially be used as the initialization point of an iterative steepest descent search [22], [24] that may converge to a (suboptimal in general) spreading code with higher SINR. In this spirit, we repeated the studies of Figs. 1 and 2 and fed the returned codes $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ to a Hamming-distance-1 steepest descent search to produce the improved spreading codes $\boldsymbol{s}_{1,\mathrm{SD}}$ and $\boldsymbol{s}_{2,\mathrm{SD}}$, respectively. In Fig. 3, we plot the SINR loss of $\boldsymbol{s}_{1,\mathrm{SD}}$ and $\boldsymbol{s}_{2,\mathrm{SD}}$; in Fig. 4, we plot the corresponding probabilities of full-rank optimality. The SINR superiority of the rank-2 approach is maintained. Interestingly, we observe that $\Pr\{\boldsymbol{s}_{2,\mathrm{SD}} = \boldsymbol{s}_{\mathrm{OPT}}\} \geq 0.9$ for any interference load. For example, when $K = 20$ the probability of global optimality is 0.95 for $\boldsymbol{s}_{2,\mathrm{SD}}$ (and 0.79 for $\boldsymbol{s}_{1,\mathrm{SD}}$). To examine the speed of convergence of the steepest descent search, in Fig. 5 we plot the evolution of the SINR value as a function of the number of iterations. The SINR of the full-rank-optimal code $\boldsymbol{s}_{\mathrm{OPT}}$ is included as reference. We observe that when the search is initialized at the proposed rank-2 code, convergence is achieved within a couple of iterations.

V. CONCLUSION

In the broad context of spread-spectrum communications (for example, CDMA systems) or chip-based signal waveform design, we considered the problem of adaptively identifying the

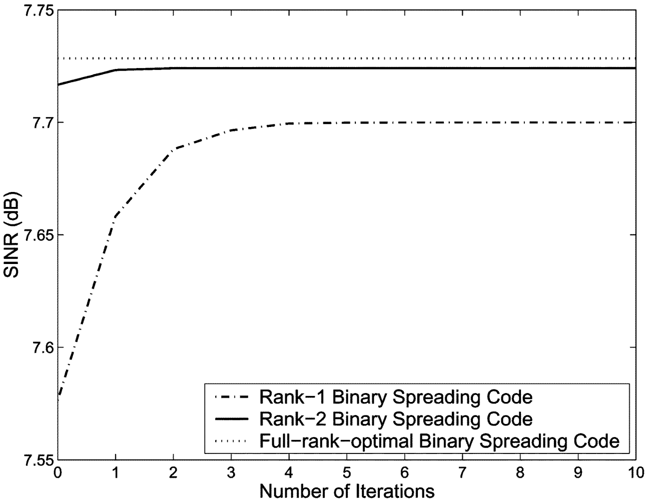

Fig. 5. SINR of steepest descent with rank-1 and rank-2 binary code initialization versus number of iterations.

binary code that maximizes the SINR at the output of the maximum-SINR filter. The optimal code is a function of the disturbance autocovariance matrix and the optimization problem is NP-hard. Instead, we developed a new algorithm of less than quadratic complexity for the computation of the optimal code under rank-2 approximation of the inverse disturbance autocovariance matrix.

As an illustration of practical significance, we demonstrated the great SINR performance improvement of the proposed rank-2 adaptive design for overloaded CDMA systems with unequal user power over the rank-1-optimal adaptive design which—as we showed—is equivalent to direct hard-limiting of the minimum-eigenvalue eigenvector of the disturbance autocovariance matrix. In fact, the proposed rank-2 design practically doubles the probability that the returned code is full-rank-optimal and when the rank-2 design is used to initialize trivial Hamming-distance-1 steepest descent, the full-rank optimal code is reached with probability greater than 0.9 (within three iterations). Certainly, the proposed adaptive binary code design may serve well as the initialization point for other, potentially more sophisticated, iterative binary search procedures.

REFERENCES

[1] P. Viswanath, V. Anantharam, and D. N. C. Tse, “Optimal sequences, power control, and user capacity of synchronous CDMA systems with linear mmse multiuser receivers,” *IEEE Trans. Inf. Theory*, vol. 45, no. 6, pp. 1968–1983, Sep. 1999.

[2] P. Cotae, “An algorithm for obtaining Welch bound equality sequences for S-CDMA channels,” *AEÜ. Int. J. Electron. Commun.*, vol. 55, pp. 95–99, Mar. 2001.

[3] T. Guess, “User-capacity-maximization in synchronous CDMA subject to RMS-bandlimited signature waveforms,” *IEEE Trans. Commun.*, vol. 52, no. 3, pp. 457–466, Mar. 2004.

[4] S. Stańczak and H. Boche, “Optimal signature sequences for asynchronous CDMA systems with fixed signal delays,” in *Proc. 5th Int. Symp. Wireless Personal Multimedia Commununication*, Honolulu, HI, Oct. 2002, vol. 3, pp. 1147–1151.

[5] H. Boche and S. Stańczak, “Iterative algorithm for finding optimal resource allocation in symbol asynchronous CDMA channels with different SINR requirements,” in *Proc. 36th Asilomar Conf. Signals, Systems and Computers*, Pacific Grove, CA, Nov. 2002, vol. 2, pp. 1909–1913.

[6] J. Luo, S. Ulukus, and A. Ephremides, “Optimal sequences and sum capacity of symbol asynchronous CDMA systems,” *IEEE Trans. Inf. Theory*, vol. 51, no. 8, pp. 2760–2769, Aug. 2005.

[7] P. B. Rapajic and B. S. Vucetic, “Linear adaptive transmitter-receiver structures for asynchronous CDMA systems,” *Europ. Trans. Telecommun.*, vol. 6, pp. 21–28, Jan.–Feb. 1995.

[8] T. F. Wong and T. M. Lok, “Transmitter adaptation in multicode DS-CDMA systems,” *IEEE J. Select. Areas Commun.*, vol. 19, no. 1, pp. 69–82, Jan. 2001.

[9] S. Ulukus and R. D. Yates, “Iterative construction of optimum signature sets in synchronous CDMA systems,” *IEEE Trans. Inf. Theory*, vol. 47, no. 5, pp. 1989–1998, Jul. 2001.

[10] C. Rose, S. Ulukus, and R. D. Yates, “Wireless systems and interference avoidance,” *IEEE Trans. Wireless Commun.*, vol. 1, pp. 415–428, Jul. 2002.

[11] S. Ulukus and R. D. Yates, “User capacity of asynchronous CDMA systems with matched filter receivers and optimum signature sequences,” *IEEE Trans. Inf. Theory*, vol. 50, no. 5, pp. 903–909, May 2004.

[12] J. L. Concha and S. Ulukus, “Optimization of CDMA signature sequences in multipath channels,” in *Proc. IEEE Vehicular Technology Conf.*, Rhodes, Greece, May 2001, vol. 3, pp. 1978–1982.

[13] O. Popescu and C. Rose, “Sum capacity and TSC bounds in collaborative multibase wireless systems,” *IEEE Trans. Inf. Theory*, vol. 50, no. 10, pp. 2433–2438, Oct. 2004.

[14] P. Cotae, “On minimizing total weighted square correlation of CDMA systems,” in *Proc. IEEE Int. Symp. Information Theory*, Yokohama, Japan, Jul. 2003, p. 389.

[15] J. L. Massey and T. Mittelholzer, “Welch’s bound and sequence sets for code-division multiple-access systems,” in *Sequences II, Methods in Communication, Security, and Computer Sciences*, R. Capocelli, A. De Santis, and U. Vaccaro, Eds. New York: Springer-Verlag, 1993, pp. 63–78.

[16] M. Rupf and J. L. Massey, “Optimum sequence multisets for synchronous code-division multiple-access channels,” *IEEE Trans. Inf. Theory*, vol. 40, no. 4, pp. 1261–1266, Jul. 1994.

[17] G. N. Karystinos and D. A. Pados, “New bounds on the total squared correlation and optimum design of DS-CDMA binary signature sets,” *IEEE Trans. Commun.*, vol. 51, no. 1, pp. 48–51, Jan. 2003.

[18] C. Ding, M. Golin, and T. Kløve, “Meeting the Welch and Karystinos-Pados bounds on DS-CDMA binary signature sets,” in *Des., Codes Cryptogr.*, Aug. 2003, vol. 30, pp. 73–84.

[19] V. P. Ipatov, “On the Karystinos–Pados bounds and optimal binary DS-CDMA signature ensembles,” *IEEE Commun. Lett.*, vol. 8, no. 2, pp. 81–83, Feb. 2004.

[20] G. N. Karystinos and D. A. Pados, “The maximum squared correlation, total asymptotic efficiency, and sum capacity of minimum total-squared-correlation binary signature sets,” *IEEE Trans. Inf. Theory*, vol. 51, no. 1, pp. 348–355, Jan. 2005.

[21] P. D. Papadimitriou and C. N. Georghiades, “Code-search for optimal TSC binary sequences with low crosscorrelation spectrum,” in *Proc. IEEE MILCOM*, Boston, MA, Oct. 2003, vol. 2, pp. 1071–1076.

[22] F. Vanhaverbeke and M. Moeneclaey, “Binary signature sets for increased user capacity on the downlink of CDMA systems,” *IEEE Trans. Wireless Commun.*, vol. 5, no. 7, pp. 1795–1804, Jul. 2006.

[23] G. N. Karystinos and D. A. Pados, “Adaptive assignment of binary user spreading codes in DS-CDMA systems,” in *Proc. SPIE 15th Annu. Int. Symp., Digital Wireless Communications Conf.*, Orlando, FL, Apr. 2001, vol. 4395, pp. 137–144.

[24] C. W. Sung and H. Y. Kwan, “Heuristic algorithms for binary sequence assignment in DS-CDMA systems,” in *Proc. IEEE Int. Symp. Personal, Indoor and Mobile Radio Communication*, Lisbon, Portugal, Sep. 2002, vol. 5, pp. 2327–2331.

[25] K. M. Mackenthun, Jr, “A fast algorithm for multiple-symbol differential detection of MPSK,” *IEEE Trans. Commun.*, vol. 42, no. 2/3/4, pp. 1471–1474, Feb./Mar./Apr. 1994.

[26] W. Sweldens, “Fast block noncoherent decoding,” *IEEE Commun. Lett.*, vol. 5, no. 4, pp. 132–134, Apr. 2001.

[27] P. Anigstein and V. Anantharam, “Ensuring convergence of the MMSE iteration for interference avoidance to the global optimum,” *IEEE Trans. Inf. Theory*, vol. 49, no. 4, pp. 873–885, Apr. 2003.

[28] C. Rose, “CDMA codeword optimization: Interference avoidance and convergence via class warfare,” *IEEE Trans. Inf. Theory*, vol. 47, no. 6, pp. 2368–2382, Sep. 2001.