Introduction

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Signal Constellations

Signal Constellations and Signal Energy

- We define a finite signal constellation as:

$$\mathcal{S} = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^n, \quad n \in \mathbb{Z}$$

  Each $\mathbf{x}$ is called a signal and has a dimensionality of n.
- $\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \{x_i\}_{i=1}^{n} \in \mathbb{R}$

Information Rate

- For simplicity, let $|\mathcal{S}| = M = 2^m, \quad m \in \mathbb{Z}^+$
  Then the maximum information carried by any $\mathbf{x}$ is:

$$m = \log_2 |\mathcal{S}| = \log_2 M \text{ bits}$$

- If the signal rate is $\frac{1}{T}$, where T is the signal duration, then the data rate is:

$$R_b = \frac{m}{T} = \frac{\log_2 M}{T} \text{ bit/s}$$

Distance metrics

- Euclidean distance $d_E(\mathbf{x}, \mathbf{x}')$:

$$d_E(\mathbf{x}, \mathbf{x}') = ||\mathbf{x} - \mathbf{x}'|| = \sqrt{\sum_{i=1}^{n} ||x_i - x_i'||^2}$$

- Hamming distance $d_H(\mathbf{x}, \mathbf{x}')$:

The number of components in which the two vectors differ.

For example:

$$\mathbf{x}_1 = (1, -1, -1, 1),\ \mathbf{x}_2 = (1, 1, -1, -1)$$

Then:

$$d_H(\mathbf{x}_1, \mathbf{x}_2) = 2$$

Choosing a modulation scheme

Bandwidth Occupancy

- The Shannon bandwidth of an N-dimensional signal set is defined as:

$$W = \frac{N}{2T} \text{ Hz}$$

- Shannon bandwidth: the minimum bandwidth that the signal **needs**.
- Fourier bandwidth: the bandwidth that the signal actually **uses**.

Signal-to-Noise Ratio (1/2)

- We define the signal energy as the norm:

$$||\mathbf{x}||^2 = x_1^2 + x_2^2 + \ldots + x_n^2$$

  and the average signal energy of a constellation as:

$$\mathcal{E} = \frac{1}{M} \sum_{\mathbf{x} \in \mathcal{S}} ||\mathbf{x}||^2$$

- Since each symbol carries (at most) $\log_2 M$ bits, we can define the average energy per bit as:

$$\mathcal{E}_b = \frac{\mathcal{E}}{\log_2 M}$$

Signal-to-Noise Ratio (2/2)

- The average power expended by the modulator is:

$$\mathcal{P} = \frac{\mathcal{E}}{T} = \mathcal{E}_b \frac{\log_2 M}{T} = \mathcal{E}_b R_b$$

- Average noise power is defined as:

$$\mathcal{P}_n = \frac{N_o}{2} \cdot 2W = N_o W$$

where W is the Shannon bandwidth of the signal.

- The Signal-to-Noise ratio (SNR) is the ratio between the average signal power and the average noise power.

$$\text{SNR} \triangleq \frac{\mathcal{P}}{\mathcal{P}_n} = \frac{\mathcal{E}_b}{N_o} \frac{R_b}{W}$$

Bandwidth Efficiency and Asymptotic Power Efficiency

- The ratio R_b/W is called the bandwidth efficiency of a modulation scheme. The higher the ratio, the better the scheme makes use of the available bandwidth W.
- We define the asymptotic power efficiency as:

$$\gamma \triangleq \frac{d_{E,min}^2}{4\mathcal{E}_b}$$

- The asymptotic power efficiency (γ) expresses how efficiently a constellation makes use of the available energy to achieve a given minimum Euclidean distance between its points.

Error Probability

Error Probability (1/5)

- In general, the received signal is a distorted version of the transmitted signal. Thus, we introduce the symbol error probability, which is the probability $P(e)$ that the demodulator will make a wrong estimation ($\hat{\mathbf{x}}$) of the transmitted symbol ($\mathbf{x}$) based on the received symbol, which is defined as follows:

$$P(e) \triangleq \frac{1}{M} \sum_{\mathbf{x}} \mathbb{P}(\hat{\mathbf{x}} \neq \mathbf{x} | \mathbf{x})$$

- Since one symbol error produces at least one bit error and at most $\log_2 M$ bit errors, a simple bound for the bit error probability P_b (also called Bit Error Rate - BER) is:

$$\frac{P(e)}{\log_2 M} \leq P_b \leq P(e)$$

Error Probability (2/5)

- We define the Voronoi (or decision) region for $\mathbf{x} \in \mathcal{S}$ as:

$$\mathcal{R}(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : ||\mathbf{y} - \mathbf{x}|| = \min_{\mathbf{x}' \in \mathcal{S}} ||\mathbf{y} - \mathbf{x}'||\}$$

- The probability of an erroneous demodulation when $\mathbf{x}$ is transmitted is given by:

$$\begin{aligned} P(e|\mathbf{x}) &= \mathbb{P}[\mathbf{y} \notin \mathcal{R}(\mathbf{x})|\mathbf{x}] \\ &= 1 - \mathbb{P}[\mathbf{y} \in \mathcal{R}(\mathbf{x})|\mathbf{x}] \end{aligned}$$

- The above expression is generally hard to compute, so it is useful to introduce an upper bound to the error probability.

Error Probability (3/5)

- We define the pairwise error probability $P(\mathbf{x} \to \hat{\mathbf{x}})$ as the probability that, when $\mathbf{x}$ is transmitted, $\hat{\mathbf{x}}$ is received.
- $P(e|\mathbf{x})$ can be expressed as the probability that at least one $\hat{\mathbf{x}} \neq \mathbf{x}$ is closer than $\mathbf{x}$ to $\mathbf{y}$.
- Using the upper bound to the probability of a union of events, we can write:

$$P(e|\mathbf{x}) \leq \sum_{\hat{\mathbf{x}} \neq \mathbf{x}} P(\mathbf{x} \to \hat{\mathbf{x}})$$

- Finally:

$$P(e) = \frac{1}{M} \sum_{\mathbf{x} \in \mathcal{S}} P(e|\mathbf{x}) \leq \frac{1}{M} \sum_{\mathbf{x} \in \mathcal{S}} \sum_{\hat{\mathbf{x}} \neq \mathbf{x}} P(\mathbf{x} \to \hat{\mathbf{x}})$$

Error Probability (4/5)

- For the simple case of the AWGN channel:

$$\mathbf{y} = \mathbf{x} + \mathbf{z}, \quad \mathbf{z} \sim \mathcal{N}(\mathbf{0}, \frac{N_o}{2} I_n)$$

- The PEP can be computed in closed form as follows:

$$\begin{aligned} P(\mathbf{x} \to \hat{\mathbf{x}}) &= \mathbb{P}(||\mathbf{y} - \hat{\mathbf{x}}||^2 < ||\mathbf{y} - \mathbf{x}||^2 | \mathbf{x}) \\ &= \mathbb{P}(||(\mathbf{x} + \mathbf{z}) - \hat{\mathbf{x}}||^2 < ||(\mathbf{x} + \mathbf{z}) - \mathbf{x}||^2) \\ &= \mathbb{P}(||(\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{z}||^2 < ||\mathbf{z}||^2) \\ &= \mathbb{P}(||\mathbf{x} - \hat{\mathbf{x}}||^2 + ||\mathbf{z}||^2 + 2(\mathbf{z}, \mathbf{x} - \hat{\mathbf{x}}) < ||\mathbf{z}||^2) \\ &= \mathbb{P}(||\mathbf{x} - \hat{\mathbf{x}}||^2 < 2(\mathbf{z}, \mathbf{x} - \hat{\mathbf{x}})) \\ &= \mathbb{P}(||\mathbf{x} - \hat{\mathbf{x}}||^2/2 < (\mathbf{z}, \mathbf{x} - \hat{\mathbf{x}})) \end{aligned}$$

- $(\mathbf{z}, \mathbf{x} - \hat{\mathbf{x}})$ is a Gaussian RV with mean 0 and variance $N_o||\mathbf{x} - \hat{\mathbf{x}}||^2/2$.

Error Probability (5/5)

- We know that for a zero mean Gaussian RV it holds that:

$$P(X > x) = Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{+\infty} e^{-\frac{x^2}{2}}$$

- So, we have:

$$P(\mathbf{x} \to \hat{\mathbf{x}}) = Q(\frac{||\mathbf{x} - \hat{\mathbf{x}}||^2}{2} \cdot \sqrt{\frac{2}{N_o||\mathbf{x} - \hat{\mathbf{x}}||^2}}) = Q(\frac{||\mathbf{x} - \hat{\mathbf{x}}||}{\sqrt{2N_o}})$$

- Using the Bhattacharyya bound:

$$Q(x) \leq e^{-x^2/2}, \quad x \geq 0$$

we can derive the following approximation:

$$P(\mathbf{x} \to \hat{\mathbf{x}}) \leq e^{-\frac{||\mathbf{x} - \hat{\mathbf{x}}||^2}{4N_o}}$$

Geometrically Uniform Constellations (1/2)

- An isometry of $\mathbb{R}^n$ is a transformation $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ that preserves Euclidean distances:

$$||u(\mathbf{x}) - u(\mathbf{y})|| = ||\mathbf{x} - \mathbf{y}||$$

- An isometry u that leaves $\mathcal{S}$ invariant, such that

$$u(\mathcal{S}) = \mathcal{S}$$

  is called a symmetry of $\mathcal{S}$. Obviously, each symmetry is an isometry.

- $\mathcal{S}$ is geometrically uniform if, given any two points $x_i, x_j \in \mathcal{S}$, there exists a symmetry $u_{i \rightarrow j}(x_i) = x_j$

Geometrically Uniform Constellations (2/2)

- All geometric properties of a GU constellation $\mathcal{S}$ relative to some point in it, do not depend on which point is chosen.
- The PEP is generally not independent of the $\mathbf{x}$ under consideration, unless we choose a GU constellation, thus easing the calculation of the error probability bound and the exact error probability.
- So, it holds that:

$$P(e) = P(e|\mathbf{x}) \leq \sum_{\hat{\mathbf{x}} \neq \mathbf{x}} Q(\frac{||\mathbf{x} - \hat{\mathbf{x}}||}{\sqrt{2N_o}})$$