provide $H_p(u) < (1/2)\ln(N)$ for $p < (1/2)$ and one basis set that will provide $H_p(u) > (1/2)\ln(N)$ for $p > (1/2)$;

2) an orthogonal basis where all basis functions are such that $H_p(u) = (1/2)\ln(N)$ for all $0 \le p \le 1$.

Some work in the first method (the over-complete basis) has been performed. Our conjecture is consistent with all known experimental results in this area. The reader is advised to examine some of the results in our original paper [5] to trace this history. However, until now, we had not been able to find a basis such as that suggested in the second method. With our new knowledge, we have been able construct an orthonormal basis for $\boldsymbol{C}^N$, for $N = K^2$, with $H_{1/2}(u)$ minimal and $H_p(u)$ independent of p for all $0 \le p \le 1$ for every basis vector u. Thus, these basis vectors are optimally localized in the phase plane and do not favor either time or frequency. Consequently, they should work well for all kinds of signals. Moreover, the associated transform can be implemented at least as fast as the fast Fourier transform of the same length. Details and applications are forthcoming [6].

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Least-Squares Channel Equalization Performance versus Equalization Delay

Athanasios P. Liavas

***Abstract*—Linear channel equalization has been a successful way to combat intersymbol interference (ISI) introduced by physical communication channels at high enough symbol rates. We consider the performance of least-squares equalizers in the single-input/multi-output (SIMO) channel context when the true channel is composed of an mth-order significant part and tails of "small" leading and/or trailing terms. Using a perturbation analysis approach, we show that if the diversity of the significant part is sufficiently large with respect to the size of the tails, then the lth-order least-squares equalizers, with $l \ge m - 1$, perform well for all the delays corresponding to the significant part. On the other hand, the performance of the equalizers for the delays corresponding to the tails may be poor.**

***Index Terms*—Communications, multichannel system identification.**

Manuscript received March 12, 1999; revised December 23, 1999. This work was supported by the T.M.R. Program of the European Commission, under Contract ERBFMBICT982878 and by EPETII of the Greek Secretariat of Research and Technology. The associate editor coordinating the review of this paper and approving it for publication was Prof. Lang Tong.

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Publisher Item Identifier S 1053-587X(00)04074-5.

I. INTRODUCTION

It is well known that in the single-input/multi-output (SIMO) channel setting, which is derived either by oversampling the channel or by using an array of sensors at the receiver, if the subchannels do not share common zeros, then an Lth-order multichannel equalizer can equalize perfectly an Mth-order noiseless channel, with $L \ge M - 1$ [1]. A case commonly encountered in practice is when the Mth-order true subchannels possess a significant part of order m, with $m < M$, and tails of "small" leading and/or trailing impulse response terms [2]. Implementation cost considerations force us to investigate which is the smallest possible order that an equalizer should have, in this case, in order to offer acceptable performance. To our knowledge, there does *not* exist a theoretical answer to this question. Furthermore, especially in the SIMO channel context, *no* theoretical explanation has been given to the fact that for some delays, equalization performance appears inherently poor, whereas for some others, it is usually satisfactory [3].

We consider the least-squares (LS) equalization of noiseless SIMO channels in the cases in which the Mth-order true subchannels possess a significant part of order m and tails of "small" leading and/or trailing terms. Using a perturbation analysis approach, we show the following.

1) If the diversity of the significant part is sufficiently large with respect to the size of the tails, then the lth-order LS equalizers, with $l \ge m - 1$, attempting to equalize the Mth-order true channel, offer good performance for *all* the delays corresponding to the significant part.

2) The performance of the LS equalizers for the delays corresponding to the tails may be poor.

II. LS SIMO CHANNEL EQUALIZATION

We consider the single-input/two-output channel setting, resulting either by oversampling the channel by a factor of 2 or by using two sensors at the receiver. Extension of our results to the single-input/p-output setting, with $p > 2$, is straighforward. If the true channel order is M, then the output of the jth subchannel $x_n^{(j)}$ for $j = 1, 2$ is given by $x_n^{(j)} = \sum_{k=0}^{M} h_k^{(j)} s_{n-k}$, where $h_k^{(j)}$ is the impulse response of the jth subchannel, and s_n is the input sequence. We denote the impulse response of the jth subchannel for $j = 1, 2$, as $\mathbf{h}_M^j \triangleq [h_0^{(j)} \ \cdots \ h_M^{(j)}]^T$, where superscript ^{T} denotes transpose, and the entire channel parameter vector is denoted by $\mathbf{h}_M \triangleq [\mathbf{h}_M^{1T} \ \mathbf{h}_M^{2T}]^T$. By stacking the $(L+1)$ most recent samples of each subchannel, we construct the data vector $\mathbf{x}_L(n) \triangleq [x_n^{(1)} \ \cdots \ x_{n-L}^{(1)} \ x_n^{(2)} \ \cdots \ x_{n-L}^{(2)}]^T$, which can be expressed as $\mathbf{x}_L(n) = \mathcal{H}_L(\mathbf{h}_M)\mathbf{s}_{L+M}(n)$, where the $2(L+1) \times (L+M+1)$ filtering matrix $\mathcal{H}_L(\mathbf{h}_M)$ is defined as

$$\mathcal{H}_L(\mathbf{h}_M) \triangleq \begin{bmatrix} \mathcal{F}_L(\mathbf{h}_M^1) \\ \mathcal{F}_L(\mathbf{h}_M^2) \end{bmatrix}$$

$$\mathcal{F}_L(\mathbf{h}_M^i) \triangleq \begin{bmatrix} h_0^{(i)} & \cdots & \cdots & h_M^{(i)} & & \\ & \ddots & & & \ddots & \\ & & h_0^{(i)} & \cdots & \cdots & h_M^{(i)} \end{bmatrix}$$

and $\mathbf{s}_{L+M}(n) \triangleq [s_n \ \cdots \ s_{n-L-M}]^T$. It is well established that if $L \ge M - 1$ and subchannels $\mathbf{h}_M^1$ and $\mathbf{h}_M^2$ do not share common zeros, then $\mathcal{H}_L(\mathbf{h}_M)$ is of full-column rank, i.e., rank $(\mathcal{H}_L(\mathbf{h}_M)) = L+M+1$. This means that the canonical vectors $\mathbf{e}_d$, that is, the vectors with 1 at the dth position and zeros elsewhere, for $d = 1, \ldots, L + M + 1$, belong to the range space of $\mathcal{H}_L^T(\mathbf{h}_M)$. As a consequence, the multichannel equalizer defined by $\mathbf{g}_{L,d} \triangleq (\mathcal{H}_L^T(\mathbf{h}_M))^\sharp \, \mathbf{e}_d$, where superscript $^\sharp$ denotes the Moore–Penrose generalized inverse, equalizes perfectly channel $\mathbf{h}_M$ for delay $(d-1)$.

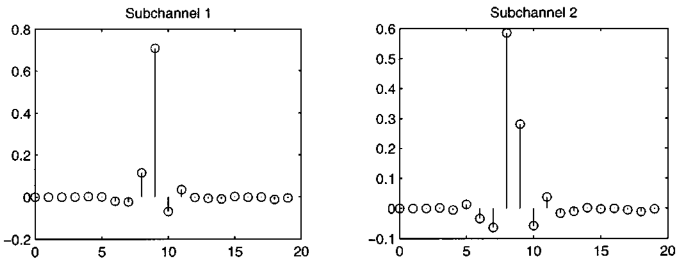

Fig. 1. Portion of the real part of the subchannels of *chan2.mat*.

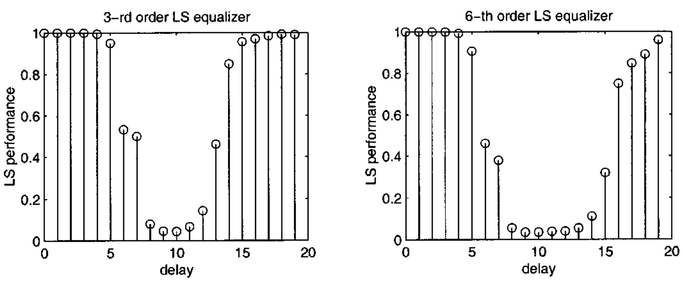

Fig. 2. Two-norm of residuals of lth-order LS equalizers for delays $\{0, \ldots, 19\}$ for *chan2.mat* ($l = 3, 6$).

In the sequel, we assume that we know *a priori* that the subchannels of $\mathbf{h}_M$ are composed of a significant part of order m lying between positions m_1 and m_2, i.e., $m = m_2 - m_1$, and "small" tails occupying the rest of the positions, and we perform a theoretical analysis of the behavior of lth-order equalizers, with $m - 1 \leq l < M - 1$. By using lth-order equalizers, we *cannot*, in general, equalize perfectly $\mathbf{h}_M$. The best we can do, in the LS sense, is to compute the lth-order LS equalizers $\mathbf{g}_{l,d} = (\mathbf{H}_l^T(\mathbf{h}_M))^\sharp \mathbf{e}_d$ for $d = 1, \ldots, l + M + 1$. It turns out that their performance is strongly dependent on the choice of d. We illustrate this point in Figs. 1 and 2.

In Fig. 1, we plot a portion of the real part of the two subchannels constructed by the oversampled, by a factor of two, complex-valued microwave radio channel *chan2.mat*, which is found at http://spib.rice.edu/spib/microwave.html. In Fig. 2, we plot the vector 2-norm of the residuals of the lth-order LS equalizers $\|\mathbf{e}_d - \mathcal{H}_l^T(\mathbf{h}_M)\mathbf{g}_{l,d}\|_2$ for $d = 1, \ldots, 20$ and $l = 3, 6$. We observe that for certain delays, the LS performance is satisfactory, whereas for delays outside a specific range, it is not. In addition, we observe that the performance of the sixth-order LS equalizer is satisfactory for more delays than that of the third-order LS equalizer.

III. LS Equalization Performance versus Delay

Our "real-world" problem is the assessment of the performance of the LS solution of the equation $\mathcal{H}_l^T(\mathbf{h}_M)\mathbf{g}_{l,d} = \mathbf{e}_d$ for $m - 1 \leq l < M - 1$ and $d = 1, \ldots, l + M + 1$. From the dimensions of the $(l + M + 1) \times 2\,(l + 1)$ matrix $\mathcal{H}_l^T(\mathbf{h}_M)$ and the range of values of l, we obtain that rank $(\mathcal{H}_l^T(\mathbf{h}_M)) \leq 2\,(l + 1)$. This gives us the fact that out of the $(l + M + 1)$ different canonical vectors corresponding to the $(l + M + 1)$ different possible delays, at most $2(l + 1)$ may lie into or close the range space of $\mathcal{H}_l^T(\mathbf{h}_M)$. Thus, the greatest number of delays for which we may expect sufficiently good LS equalization, with an equalizer of order l, is $2(l + 1)$.

Toward developing a theoretical analysis of the performance of the lth-order LS equalizers attempting to equalize $\mathbf{h}_M$, we first partition $\mathbf{h}_M$, similarly to [7, Eqs. (3)–(6)], as $\mathbf{h}_M = \mathbf{h}_{m_1,m_2}^z + \mathbf{d}_{m_1,m_2}^z$, where $\mathbf{h}_{m_1,m_2}^z$ and $\mathbf{d}_{m_1,m_2}^z$ denote the appropriately zero-padded mth-order significant part and tails, respectively; we denote the truncated mth-order significant part by $\mathbf{h}_{m_1,m_2}$ (see [7, Eq. (7)]). We assume, without loss of generality, that $\|\mathbf{h}_M\|_2 = 1$, and we express the fact that the tails are "small" with respect to the significant part as

$$\left\|\mathbf{d}_{m_1,m_2}^z\right\|_2 = \epsilon_m \ll 1. \tag{1}$$

Then, using this partitioning, we decompose our "real-world" problem into an "ideal" problem and a perturbation, which fulfill the following conditions.

1) The "ideal" problem has a well-defined and informative solution.
2) The perturbation is "small" with respect to the "ideal" quantities.

Finally, using invariant subspace perturbation results, we assess the performance of the lth-order LS equalizers, attempting to equalize the true channel $\mathbf{h}_M$.

A. Delays Corresponding to the Significant Part

We first consider the performance of the lth-order LS equalizers for the delays corresponding to the significant part of the channel. Our analysis is performed in three steps.

In the first step, we assume that our channel is $\mathbf{h}_{m_1,m_2}$, i.e., the truncated mth-order significant part of the true channel. If $\mathbf{h}_{m_1,m_2}^1$ and $\mathbf{h}_{m_1,m_2}^2$ do not share common zeros, then $\mathcal{H}_l(\mathbf{h}_{m_1,m_2})$ is of full-column rank, i.e., $\text{rank}(\mathcal{H}_l(\mathbf{h}_{m_1,m_2})) = l + m + 1$. Thus, $\mathcal{H}_l^T(\mathbf{h}_{m_1,m_2})$ is of full-row rank, giving that $\mathcal{R}(\mathcal{H}_l^T(\mathbf{h}_{m_1,m_2})) = \mathcal{R}(\mathbf{I}_{l+m+1})$, where $\mathcal{R}(\mathcal{A})$ denotes the range space of matrix $\mathcal{A}$, and $\mathbf{I}_n$ denotes the n-dimensional identity matrix. This means that the canonical vectors $\mathbf{e}_d$, for $d = 1, \ldots, l + m + 1$, belong to the range space of $\mathcal{H}_l^T(\mathbf{h}_{m_1,m_2})$. Consequently, equation $\mathcal{H}_l^T(\mathbf{h}_{m_1,m_2})\mathbf{g}_{l,d} = \mathbf{e}_d$ always has a solution, yielding that channel $\mathbf{h}_{m_1,m_2}$ can be equalized perfectly by an lth-order equalizer, with $l \geq m - 1$. The minimum norm solution is given by $\mathbf{g}_{l,d} = (\mathcal{H}_l^T(\mathbf{h}_{m_1,m_2}))^\sharp \mathbf{e}_d$.

In the second step, we assume that our channel is $\mathbf{h}_{m_1,m_2}^z$, i.e., the appropriately zero-padded version of the mth-order significant part of the true channel. It is easy to see that

$$\text{rank}\left(\mathcal{H}_l^T\left(\mathbf{h}_{m_1,m_2}^z\right)\right) = l + m + 1 \quad \text{and}$$

$$\mathcal{R}\left(\mathcal{H}_l^T\left(\mathbf{h}_{m_1,m_2}^z\right)\right) = \mathcal{R}\left(\begin{bmatrix} \mathbf{O}_{m_1,l+m+1} \\ \mathbf{I}_{l+m+1} \\ \mathbf{O}_{M-m_2,l+m+1} \end{bmatrix}\right)$$

where $\mathbf{O}_{n,m}$ denotes the $(n \times m)$ zero matrix. This means that $\mathcal{R}(\mathcal{H}_l^T(\mathbf{h}_{m_1,m_2}^z))$ contains the canonical vectors $\mathbf{e}_d$ for $d = m_1 + 1, \ldots, m_2 + l + 1$. For the corresponding delays, $\mathbf{h}_{m_1,m_2}^z$ can be equalized perfectly by the minimum norm equalizers $\mathbf{g}_{l,d} = (\mathcal{H}_l^T(\mathbf{h}_{m_1,m_2}^z))^\sharp \mathbf{e}_d$.

Finally, in order to study the performance of the lth-order LS equalizers that attempt to equalize $\mathbf{h}_M$, we consider $\mathcal{H}_l^T(\mathbf{h}_M)$ as the result of the perturbation $\mathcal{H}_l^T(\mathbf{d}_{m_1,m_2}^z)$ acting on $\mathcal{H}_l^T(\mathbf{h}_{m_1,m_2}^z)$. We denote the matrix 2-norm of the perturbation as $\mathcal{E}_l^{m_1,m_2} \triangleq \|\mathcal{H}_l^T(\mathbf{d}_{m_1,m_2}^z)\|_2$. In order to relate $\mathcal{E}_l^{m_1,m_2}$ to the size of the tails, we use the structure of $\mathcal{H}_l^T(\mathbf{d}_{m_1,m_2}^z)$ and (1) to obtain $\|\mathcal{H}_l^T(\mathbf{d}_{m_1,m_2}^z)\|_F = \sqrt{l+1}\,\epsilon_m$, where $\|\cdot\|_F$ denotes the matrix Frobenious norm. Then, using the matrix 2-norm/Frobenious-norm inequalities [4, p. 57 and 72], we obtain

$$\begin{aligned} \frac{1}{\sqrt{2}}\epsilon_m = \frac{1}{\sqrt{2(l+1)}}\left\|\mathcal{H}_l^T(\mathbf{d}_{m_1,m_2}^z)\right\|_F &\leq \mathcal{E}_l^{m_1,m_2} \\ &\leq \left\|\mathcal{H}_l^T(\mathbf{d}_{m_1,m_2}^z)\right\|_F = \sqrt{l+1}\epsilon_m. \end{aligned} \tag{2}$$

The lth-order LS equalizer leads to a combined channel-equalizer impulse response $\mathbf{e}_{l,d}^{\text{LS}}$, with $\mathbf{e}_{l,d}^{\text{LS}} \in \mathcal{R}(\mathcal{H}_l^T(\mathbf{h}_M))$, that is closest, with respect to the vector 2-norm, to $\mathbf{e}_d$. In the sequel, we give the conditions under which, even if we constrain our search to a subspace of $\mathcal{R}(\mathcal{H}_l^T(\mathbf{h}_M))$, we can find a vector $\tilde{\mathbf{e}}_{l,d}$ that is "close" to $\mathbf{e}_d$; since the LS solution $\mathbf{e}_{l,d}^{\text{LS}}$ can only do better than $\tilde{\mathbf{e}}_{l,d}$, the fact that $\tilde{\mathbf{e}}_{l,d}$ is "close" to $\mathbf{e}_d$ means that the lth-order LS equalizers that attempt to equalize $\mathbf{h}_M$, for the delays corresponding to the significant part, perform "well." To this end, we denote by $\mathcal{R}(\mathcal{H}_l^T(\mathbf{h}_M))_{l+m+1}$ the $(l+m+1)$-dimensional subspace of $\mathcal{R}(\mathcal{H}_l^T(\mathbf{h}_M))$ spanned by the left singular vectors associated with the $(l+m+1)$ largest singular values of $\mathcal{H}_l^T(\mathbf{h}_M)$. Thus, $\mathcal{R}(\mathcal{H}_l^T(\mathbf{h}_M))_{l+m+1}$ may be considered to be the perturbed subspace corresponding to $\mathcal{R}(\mathcal{H}_l^T(\mathbf{h}^z_{m_1,m_2}))$.

In order to proceed, we need a measure of the distance between two linear subspaces $\mathcal{X}$ and $\mathcal{Y}$. Such a measure, which is commonly encountered in numerical analysis, is the sine of their *canonical angles*, which is denoted $\|\sin\angle(\mathcal{X},\mathcal{Y})\|_2$. It is well known that [5, p. 92]

$$\rho_{g,2}(\mathcal{X},\mathcal{Y}) = \|\sin\angle(\mathcal{X},\mathcal{Y})\|_2 \tag{3}$$

where $\rho_{g,2}(\mathcal{X},\mathcal{Y})$ is the 2-*gap* between $\mathcal{X}$ and $\mathcal{Y}$ defined as [5, p. 91]

$$\rho_{g,2}(\mathcal{X},\mathcal{Y}) \triangleq \max\left\{ \max_{\substack{x\in\mathcal{X}\\ \|x\|_2=1}} \delta_2(x,\mathcal{Y}),\ \max_{\substack{y\in\mathcal{Y}\\ \|y\|_2=1}} \delta_2(y,\mathcal{X}) \right\} \tag{4}$$

with

$$\delta_2(x,\mathcal{Y}) \triangleq \min_{y\in\mathcal{Y}} \|x-y\|_2. \tag{5}$$

The theorem that follows provides an upper bound for the distance between $\mathcal{R}(\mathcal{H}_l^T(\mathbf{h}^z_{m_1,m_2}))$ and $\mathcal{R}(\mathcal{H}_l^T(\mathbf{h}_M))_{l+m+1}$.

Theorem 1: Let $\mathcal{R}(\mathcal{H}_l^T(\mathbf{h}^z_{m_1,m_2}))$ denote the $(l+m+1)$-dimensional range space of $\mathcal{H}_l^T(\mathbf{h}^z_{m_1,m_2})$, $\sigma_{l+m+1}(\mathcal{H}_l^T(\mathbf{h}^z_{m_1,m_2}))$ denote the smallest nonzero singular value of $\mathcal{H}_l^T(\mathbf{h}^z_{m_1,m_2})$, and $\mathcal{R}(\mathcal{H}_l^T(\mathbf{h}_M))_{l+m+1}$ denote the $(l+m+1)$-dimensional subspace spanned by the left singular vectors of $\mathcal{H}_l^T(\mathbf{h}_M)$ associated with its $(l+m+1)$ largest singular values. Let $\mathcal{E}_l^{m_1,m_2}$ be the matrix 2-norm of the perturbation $\mathcal{H}_l^T(\mathbf{d}^z_{m_1,m_2})$. If $\mathcal{E}_l^{m_1,m_2} \le \sigma_{l+m+1}(\mathcal{H}_l^T(\mathbf{h}^z_{m_1,m_2}))/2$, then

$$\begin{aligned}&\left\|\sin\angle\left(\mathcal{R}\left(\mathcal{H}_l^T\left(\mathbf{h}^z_{m_1,m_2}\right)\right), \mathcal{R}\left(\mathcal{H}_l^T(\mathbf{h}_M)\right)_{l+m+1}\right)\right\|_2 \\ &\quad\le \frac{\mathcal{E}_l^{m_1,m_2}}{\sigma_{l+m+1}\left(\mathcal{H}_l^T\left(\mathbf{h}^z_{m_1,m_2}\right)\right) - \mathcal{E}_l^{m_1,m_2}}.\end{aligned} \tag{6}$$

Otherwise, the upper bound is equal to 1.

Proof: The theorem follows easily from the "*generalized* $\sin\theta$ *theorem*" of [6]. ∎

From (3)– (6) and the facts that $\mathbf{e}_d \in \mathcal{R}(\mathcal{H}_l^T(\mathbf{h}^z_{m_1,m_2}))$ for $d = m_1+1,\ldots,m_2+l+1$ and $\|\mathbf{e}_d\|_2 = 1$, we deduce that there is an $\tilde{\mathbf{e}}_{l,d} \in \mathcal{R}(\mathcal{H}_l^T(\mathbf{h}_M))_{l+m+1}$ such that

$$\|\mathbf{e}_d - \tilde{\mathbf{e}}_{l,d}\|_2 \le \frac{\mathcal{E}_l^{m_1,m_2}}{\sigma_{l+m+1}\left(\mathcal{H}_l^T\left(\mathbf{h}^z_{m_1,m_2}\right)\right) - \mathcal{E}_l^{m_1,m_2}}.$$

Using the fact that the LS solution $\mathbf{e}_{l,d}^{\mathrm{LS}}$ can only do better than $\tilde{\mathbf{e}}_{l,d}$, we obtain

$$\begin{aligned}\left\|\mathbf{e}_d - \mathbf{e}_{l,d}^{\mathrm{LS}}\right\|_2 &\le \|\mathbf{e}_d - \tilde{\mathbf{e}}_{l,d}\|_2 \\ &\le \frac{\mathcal{E}_l^{m_1,m_2}}{\sigma_{l+m+1}\left(\mathcal{H}_l^T\left(\mathbf{h}^z_{m_1,m_2}\right)\right) - \mathcal{E}_l^{m_1,m_2}}.\end{aligned} \tag{7}$$

Bound (7) is a worst-case quantity. It means that if $\sigma_{l+m+1}(\mathcal{H}_l^T(\mathbf{h}^z_{m_1,m_2}))$ is sufficiently large with respect to $\mathcal{E}_l^{m_1,m_2}$, then the lth-order LS equalizers that attempt to equalize $\mathbf{h}_M$ perform well for *all* the delays corresponding to the significant part. Assessment of the best-case performance remains a very interesting problem, especially in the cases in which $\sigma_{l+m+1}(\mathcal{H}_l^T(\mathbf{h}^z_{m_1,m_2}))$ is "small." Term $\sigma_{l+m+1}(\mathcal{H}_l^T(\mathbf{h}^z_{m_1,m_2}))$, which is the distance in the matrix 2-norm of $\mathcal{H}_l^T(\mathbf{h}^z_{m_1,m_2})$ from the matrices with rank $(l+m)$ measures "how well" fulfilled our assumption about rank $(\mathcal{H}_l^T(\mathbf{h}^z_{m_1,m_2}))$ or, equivalently, rank $(\mathcal{H}_l^T(\mathbf{h}_{m_1,m_2}))$ is. Thus, it may be interpreted as a measure of *diversity* of the significant part of the channel. For varying l, terms $\sigma_{l+m+1}(\mathcal{H}_l^T(\mathbf{h}^z_{m_1,m_2}))$ are not orderable, in general; extensive simulations have shown that they are reasonably close to each other.

Our results were derived by assuming the knowledge of the true impulse response $\mathbf{h}_M$. However, since during our analysis we used only the size and not the structure of the perturbation $\mathcal{H}_l^T(\mathbf{d}^z_{m_1,m_2})$, our results also hold for the cases in which the impulse response is known to within an $O(\epsilon_m)$ estimation error, due to, e.g., the use of training. Development of analysis by exploiting the structure of the perturbation remains a very interesting problem.

B. Delays Corresponding to the Tails

In the previous subsection, we saw that under certain conditions, the $(l+m+1)$ delays corresponding to the significant part of the channel lead to sufficiently good equalization of $\mathbf{h}_M$ by the lth-order LS equalizers for $l \ge m-1$. In addition, we saw that the dimensions of $\mathcal{H}_l^T(\mathbf{h}_M)$ imply that at most $2(l+1)$ delays may lead to sufficiently good equalization performance. This means that for equalizer order $l = m-1$, we should not expect any other delays to lead to sufficiently good equalization performance. However, for $l \ge m$, it is not immediately clear from our analysis, until now, whether there exist other delays that may lead to a sufficiently good performance or not. Thus, a natural question arises: *"Are there any other delays that may lead generically to sufficiently good LS equalization?"*

In order to answer this question, we perform a perturbation analysis similar to that of the previous subsection. However, now, in our "ideal" problem, we must consider not only the significant part of the channel but also certain "small" terms. To justify this, let us assume that we want to study the performance of the lth-order LS solution of the equation $\mathcal{H}_l^T(\mathbf{h}_M)\mathbf{g}_{l,m_1^*+1} = \mathbf{e}_{m_1^*+1}$ with $m_1^* < m_1$. If our "ideal" problem does not involve some "small" leading terms, for example, $h^{(1)}_{m_1^*}$ and $h^{(2)}_{m_1^*}$, then it is *not* possible to have a *nonzero* term at the (m_1^*+1)st position of its right-hand side. As a result, our "real-world" problem, which has 1 at the (m_1^*+1)st position of its right-hand side, can *not* be a "small," i.e., $O(\epsilon_m)$, perturbation of our "ideal" problem.

A possible "ideal" problem is

$$\mathcal{H}_l^T\left(\mathbf{h}^z_{m_1^*,m_2}\right)\mathbf{g}_{l,m_1^*+1} = \mathbf{e}_{m_1^*+1} \tag{8}$$

with $\mathbf{h}^z_{m_1^*,m_2}$ denoting the (appropriately zero padded) part of the true channel lying between indices m_1^* and m_2. The first implication of this fact is that we must consider equalizers of order $l^* \ge m^*-1$ with $m^* = m_2 - m_1^* > m$. That is, in this case, we must consider equalizers longer than the ones considered in the previous subsection. Otherwise, either a) we cannot equalize perfectly, in general, $\mathbf{h}_{m_1^*,m_2}$ or, equivalently, $\mathbf{h}^z_{m_1^*,m_2}$, thus transforming the "ideal" problem defined by (8) to a "nonideal" one, or b) we may consider some significant impulse response terms as part of the perturbation, thus decreasing the length of the "ideal" channel but increasing significantly the size of the perturbation. Both of these alternatives are clearly undesirable for the purposes of analysis.

The corresponding perturbation is $\mathcal{H}_{l^*}^T(\mathbf{d}^z_{m_1^*,m_2})$. Continuing similarly to the analysis of the previous subsection, we obtain that the key terms are $\mathcal{E}_{l^*}^{m_1^*,m_2}$, that is, the matrix 2-norm of perturbation $\mathcal{H}_{l^*}^T(\mathbf{d}^z_{m_1^*,m_2})$, and $\sigma_{l^*+m^*+1}(\mathcal{H}_{l^*}^T(\mathbf{h}^z_{m_1^*,m_2}))$, that is, the smallest nonzero singular value of $\mathcal{H}_{l^*}^T(\mathbf{h}^z_{m_1^*,m_2})$. Defining $\epsilon_{m^*} \triangleq \|\mathbf{d}^z_{m_1^*,m_2}\|_2$, we obtain [similarly to (2)]

$$\frac{1}{\sqrt{2}}\epsilon_{m^*} \le \mathcal{E}_{l^*}^{m_1^*,m_2} \le \sqrt{l^*+1}\,\epsilon_{m^*}. \tag{9}$$

Furthermore, since the small leading and/or trailing terms are usually of the same order of magnitude [2], we usually have that $\epsilon_m \approx \epsilon_{m^*}$. Using

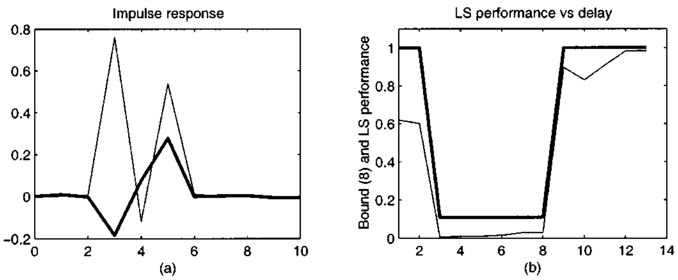

Fig. 3. (a) Tenth-order impulse response. (b) Bound (7) (thick line) and 2-norm of residuals of the third-order LS equalizers versus delay.

Theorem 1, we obtain that if $\mathcal{E}_{l^*}^{m_1^*,m_2}$ is sufficiently small with respect to $\sigma_{l^*+m^*+1}(\mathcal{H}_{l^*}^T(\mathbf{h}_{m_1^*,m_2}^z))$, then we expect $\mathbf{e}_{m_1^*+1}$ to be close to $\mathcal{R}(\mathcal{H}_{l^*}^T(\mathbf{h}_M))_{l^*+m^*+1}$, implying that the l^*th-order LS equalizer that attempts to equalize $\mathbf{h}_M$ for delay m_1^* performs well. In the sequel, we use a result of [7], which shows that $\sigma_{l^*+m^*+1}(\mathcal{H}_{l^*}^T(\mathbf{h}_{m_1^*,m_2}^z))$ becomes of the order of $\mathcal{E}_{l^*}^{m_1^*,m_2}$, leading to potentially poor performance of the l^*th-order LS equalizer for delay m_1^*.

Theorem 2: If $\sigma_{l+m+1}(\mathcal{H}_l^T(\mathbf{h}_{m_1,m_2}^z))$ denotes the smallest nonzero singular value of the rank-$(l+m+1)$ matrix $\mathcal{H}_l^T(\mathbf{h}_{m_1,m_2}^z)$, then

$$\sigma_{l+m+1}\left(\mathcal{H}_l^T\left(\mathbf{h}_{m_1,m_2}^z\right)\right) \leq \min\left\{\sqrt{\left|h_{m_1}^{(1)}\right|^2+\left|h_{m_1}^{(2)}\right|^2}, \sqrt{\left|h_{m_2}^{(1)}\right|^2+\left|h_{m_2}^{(2)}\right|^2}\right\}. \quad (10)$$

In this case, (1) and the fact that $h_{m_1^*}^{(1)}$ and $h_{m_1^*}^{(2)}$ belong to the *true* channel tails, i.e., $\mathbf{d}_{m_1,m_2}^z$, give

$$\sigma_{l^*+m^*+1}\left(\mathcal{H}_{l^*}^T\left(\mathbf{h}_{m_1^*,m_2}^z\right)\right) \leq \epsilon_m. \quad (11)$$

Relations (9) and (11) and the fact that $\epsilon_m \approx \epsilon_{m^*}$ yield that $\sigma_{l^*+m^*+1}(\mathcal{H}_{l^*}^T(\mathbf{h}_{m_1^*,m_2}^z))$ becomes $O(\epsilon_m)$, rendering our "ideal" problem very sensitive to "small" perturbations. In this case, upper bound (7) becomes (close to) 1. This means that it is *not* guaranteed that there is a vector in $\mathcal{R}(\mathcal{H}_{l^*}^T(\mathbf{h}_M))_{l^*+m^*+1}$ that is close to $\mathbf{e}_{m_1^*+1}$.

We may ask if $\mathbf{e}_{m_1^*+1}$ may be generically close to the subspace spanned by the left singular vectors of $\mathcal{H}_{l^*}^T(\mathbf{h}_M)$ corresponding to its remaining nonzero singular values. It turns out that this does *not* happen because a counterexample can be easily constructed. It can be easily seen that we can null terms $h_{m_1^*}^{(1)}$ and $h_{m_1^*}^{(2)}$ of the (m_1^*+1)-st row of $\mathcal{H}_{l^*}^T(\mathbf{h}_{m_1^*,m_2}^z)$ by adding a small, i.e., $O(\epsilon_m)$, perturbation matrix composed of terms $-h_{m_1^*}^{(1)}$ and $-h_{m_1^*}^{(2)}$ at the appropriate positions of the (m_1^*+1)st row and zeros elsewhere. This small perturbation makes $\mathbf{e}_{m_1^*+1}$ orthogonal to the range space of the resulting perturbed matrix. Of course, this perturbation does *not* have the structure of $\mathcal{H}_{l^*}^T(\mathbf{d}_{m_1^*,m_2}^z)$. However, it is very informative in our framework, in which we repeat that we use only the size and not the structure of the perturbation because it implies that for the delays corresponding to the tails, we can *not* derive a worst-case bound (significantly) smaller than 1.

Analogous arguments hold for the $d > m_2+l+1$ case.

IV. Simulations

In Fig. 3(a), we plot a 10th-order two-channel impulse response composed of a significant part of order 2, lying between positions $m_1 = 3$ and $m_2 = 5$ and tails. In Fig. 3(b), we plot the 2-norm of the residuals of the third-order LS equalizers for the various delays and bound (7) (thick line). We observe that our bound is able to predict the performance of the LS equalizers for the various delays. For some delays corresponding to the tails, the 2-norm of the residual of the LS equalizers is "close" to 1, supporting the arguments of the previous subsection.

V. Conclusion

We performed a theoretical analysis of the LS equalization performance in the cases in which the Mth-order true subchannels possess an mth-order significant part with $m < M$ and tails of "small" leading and/or trailing terms. We showed that if the diversity of the significant part is sufficiently large with respect to the size of the tails, then the lth-order LS equalizers with $l \geq m-1$ perform well for all the delays corresponding to the significant part. On the other hand, the performance of the LS equalizers for the delays corresponding to the tails may be poor. In practice, it is usually poor. Our results serve as an explanation of the behavior of LS equalizers in realistic cases [3].

Acknowledgment

The author would like to thank Prof. P. A. Regalia for useful suggestions that improved the quality of the paper.

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