The *constrained* Block-PARAFAC decomposition

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Presentation outline

- Motivations and preliminaries
- Constrained Block-PARAFAC formulation
- Some terminology and concepts
- Link: constrained Block-PARAFAC and Block-Tucker3
- Uniqueness issues
- Applications in wireless signal processing
- Concluding remarks and perspectives

Motivations and preliminaries

Acknowledgement:

To Richard Harshman for his valuable comments, suggestions and motivation on important issues of this work.

Motivations and preliminaries

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- PARAFAC: no interaction between modes, *unique* (without orthogonality constraints);
- Tucker3: complete interaction between modes, *nonunique* (rotational indeterminacy);
- Mixed PARAFAC-Tucker3 models [Bro'98]:
  $\ast$ Constrained interactions involving factors different modes (not as complete as in Tucker3 models);
  $\ast$ Arise in some wireless signal processing problems:
    - Multiantenna codings
    - Blind beamforming

Motivations and preliminaries

- Mixed PARAFAC-Tucker3 decompositions:
  - ∗ Thinking “Tucker3-wise": constrained core tensor
  - ∗ Thinking “PARAFAC-wise": constrained factor matrices
- Generalizing/combining PARAFAC and Tucker3 decompositions:
  - ∗ Decomposition in a sum of smaller Tucker3 blocks: [De Lathauwer’05]
  - ∗ Decomposition in a sum of constrained PARAFAC blocks: *Special case of [De Lathauwer’05]*

Constrained Block-PARAFAC formulation

- The decomposition in scalar form ($\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$, Q blocks):

$$x_{i_1,i_2,i_3} = \sum_{q=1}^{Q} \sum_{r_1^{(q)}=1}^{R_1^{(q)}} \sum_{r_2^{(q)}=1}^{R_2^{(q)}} a^{(q)}_{i_1,r_1^{(q)}} \, b^{(q)}_{i_2,r_2^{(q)}} \, c^{(q)}_{r_1^{(q)},r_2^{(q)},i_3}.$$

Special case of [De Lathauwer'05];

- $\{\mathbf{A}^{(q)}\} \in \mathbb{C}^{I_1 \times R_1^{(q)}}$ and $\{\mathbf{B}^{(q)}\} \in \mathbb{C}^{I_2 \times R_2^{(q)}}$, $\{\mathcal{C}^{(q)}\} \in \mathbb{C}^{R_1^{(q)} \times R_2^{(q)} \times I_3}$
- Set of matrices $\{\mathbf{C}^{(q)}\} \in \mathbb{C}^{I_3 \times R_1^{(q)} R_2^{(q)}}$ defined as:

$$[\mathbf{C}^{(q)}]_{i_3,(r_1^{(q)}-1)R_2^{(q)}+r_2^{(q)}} = c^{(q)}_{r_1^{(q)},r_2^{(q)},i_3}, \quad q = 1, \ldots, Q$$

- Factorization as a sum of constrained PARAFAC blocks [de Almeida et al.'05]:

$$\mathbf{X}_{\cdot\cdot i_3} = \sum_{q=1}^{Q} \left(\mathbf{A}^{(q)} \otimes \mathbf{1}^T_{R_2^{(q)}}\right) D_{i_3}(\mathbf{C}^{(q)}) \left(\mathbf{1}^T_{R_1^{(q)}} \otimes \mathbf{B}^{(q)}\right)^T.$$

Constrained Block-PARAFAC formulation

Interaction patterns

- Equivalences:

$$\mathbf{A}^{(q)} \otimes \mathbf{1}^T_{R_2^{(q)}} = (\mathbf{A}^{(q)} \otimes 1)(\mathbf{I}_{R_1^{(q)}} \otimes \mathbf{1}^T_{R_2^{(q)}}) = \mathbf{A}^{(q)} \underbrace{(\mathbf{I}_{R_1^{(q)}} \otimes \mathbf{1}^T_{R_2^{(q)}})}_{\boldsymbol{\Psi}^{(q)}} = \mathbf{A}^{(q)} \boldsymbol{\Psi}^{(q)};$$

$$\mathbf{1}^T_{R_1^{(q)}} \otimes \mathbf{B}^{(q)} = (1 \otimes \mathbf{B}^{(q)})(\mathbf{1}^T_{R_1^{(q)}} \otimes \mathbf{I}_{R_2^{(q)}}) = \mathbf{B}^{(q)} \underbrace{(\mathbf{1}^T_{R_1^{(q)}} \otimes \mathbf{I}_{R_2^{(q)}})}_{\boldsymbol{\Phi}^{(q)}} = \mathbf{B}^{(q)} \boldsymbol{\Phi}^{(q)};$$

- Constraint matrices:

$$\boldsymbol{\Psi}^{(q)} = \mathbf{I}_{R_1^{(q)}} \otimes \mathbf{1}^T_{R_2^{(q)}}, \qquad \boldsymbol{\Phi}^{(q)} = \mathbf{1}^T_{R_1^{(q)}} \otimes \mathbf{I}_{R_2^{(q)}}$$

- The sets $\{\boldsymbol{\Psi}^{(1)}, \ldots, \boldsymbol{\Psi}^{(Q)}\}$ and $\{\boldsymbol{\Phi}^{(1)}, \ldots, \boldsymbol{\Phi}^{(Q)}\}$ reveal the interaction patterns within the different blocks;

Constrained Block-PARAFAC formulation

- Factorization using the constraint matrices

$$\mathbf{X}_{\cdot\cdot i_3} = \sum_{q=1}^{Q} \mathbf{A}^{(q)} \mathbf{\Psi}^{(q)} D_{i_3}(\mathbf{C}^{(q)})(\mathbf{B}^{(q)} \mathbf{\Phi}^{(q)})^T.$$

- Block factor matrices:

$$\begin{aligned} \mathbf{A} &= [\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(Q)}] \in \mathbb{C}^{I_1 \times R_1} \\ \mathbf{B} &= [\mathbf{B}^{(1)}, \ldots, \mathbf{B}^{(Q)}] \in \mathbb{C}^{I_2 \times R_2} \\ \mathbf{C} &= [\mathbf{C}^{(1)}, \ldots, \mathbf{C}^{(Q)}] \in \mathbb{C}^{I_3 \times R_3}, \end{aligned}$$

$$R_1 = \sum_{q=1}^{Q} R_1^{(q)}, \quad R_2 = \sum_{q=1}^{Q} R_2^{(q)}, \quad R_3 = \sum_{q=1}^{Q} R_1^{(q)} R_2^{(q)}.$$

- Block constraint matrices:

$$\begin{aligned} \mathbf{\Psi} &= BlockDiag(\mathbf{\Psi}^{(1)} \cdots \mathbf{\Psi}^{(Q)}) \quad (R_1 \times R_3) \\ \mathbf{\Phi} &= BlockDiag(\mathbf{\Phi}^{(1)} \cdots \mathbf{\Phi}^{(Q)}) \quad (R_2 \times R_3) \end{aligned}$$

Constrained Block-PARAFAC formulation

- Compact matrix-slice form:

$$\mathbf{X}_{\cdot\cdot i_3} = \mathbf{A}\boldsymbol{\Psi} D_{i_3}(\mathbf{C})(\mathbf{B}\boldsymbol{\Phi})^T.$$

- Unfolded matrices:

$$\mathbf{X}_1 = (\mathbf{C}\diamond\mathbf{A}\boldsymbol{\Psi})(\mathbf{B}\boldsymbol{\Phi})^T, \quad \mathbf{X}_2 = (\mathbf{B}\boldsymbol{\Phi}\diamond\mathbf{C})(\mathbf{A}\boldsymbol{\Psi})^T, \quad \mathbf{X}_3 = (\mathbf{A}\boldsymbol{\Psi}\diamond\mathbf{B}\boldsymbol{\Phi})\mathbf{C}^T$$

Related to PARALIND models [Bro-Harsh-Sid'05]

- Expansion using tensor products of canonical vectors:

Goal: Justify the introduction of the constraint matrices [de Almeida et al.'06]

$$\mathbf{X}_3 = \sum_{i_1=1}^{I_1}\sum_{i_2=1}^{I_2}\sum_{i_3=1}^{I_3} x_{i_1,i_2,i_3}(\mathbf{e}_{i_1}^{(I_1)} \otimes \mathbf{e}_{i_2}^{(I_2)})\mathbf{e}_{i_3}^{(I_3)T}$$

$$\mathbf{X}_3 = \sum_{i_1=1}^{I_1}\sum_{i_2=1}^{I_2}\sum_{i_3=1}^{I_3}\sum_{q=1}^{Q}\sum_{r_1=1}^{R_1}\sum_{r_2=1}^{R_2} a_{i_1,r_1}^{(q)} b_{i_2,r_2}^{(q)} c_{r_1,r_2,i_3}^{(q)}(\mathbf{e}_{i_1}^{(I_1)} \otimes \mathbf{e}_{i_2}^{(I_2)})\mathbf{e}_{i_3}^{(I_3)T}$$

Some terminology and concepts

What can be said about constrained Block-PARAFAC ?

- Factorization of a three-way array in a sum of Q constrained PARAFAC blocks, everyone of them being a function of three component matrices $\mathbf{A}^{(q)}, \mathbf{B}^{(q)}$ and $\mathbf{C}^{(q)}$.
- Within the same PARAFAC block, it is permitted that columns of different component matrices are linearly combined to generate the three-way data.
- The interaction pattern is defined by the matrices $\mathbf{\Psi}^{(q)}$ and $\mathbf{\Phi}^{(q)}$ and may differ from block to block.
- No interaction takes place between blocks.

Some terminology and concepts

"Between-block" *versus* "Within-block" uniqueness:
[Terms coined by Harshman]

- Uniqueness concepts interpreted in two ways:

  $\ast$ *Between-block uniqueness*: synonym of *separability* of the Q blocks. Independent of the interaction structure.

  $\ast$ *Within-block uniqueness*: unique determination of the three component matrices of the corresponding block (up to permutation and scaling). Dependent of the particular interaction structure.

- Connection with "partial" uniqueness concepts:

  $\ast$ PARAFAC case [Harshman'72] [ten Berge'04] [Bro-Harsh-Sid'05]:

  "*When non-unique solutions occur ... uniqueness can partially or completely "break down"..."* [Harshman'72]

  $\ast$ Constrained Block-PARAFAC case:

  Uniqueness can break down "in parts" (within a block)... but also "between blocks"!

Constrained Block-PARAFAC *linked to* Block-Tucker3

- Constrained Block-PARAFAC can be approached to Tucker3 analysis [Kiers&Smilde'98] [ten Berge&Smilde'02]:

  Proposition: *The constrained Block-PARAFAC decomposition is equivalent to a "constrained Block-Tucker3" one, where the unfolded matrices of the every core tensor block are column-wise orthogonal, i.e., the inner product of any two distinct columns of every unfolded core matrix is equal to zero.*

- The link relies on the concept of block Khatri-Rao product $|\otimes|$: $\mathbf{A} \,|\otimes|\, \mathbf{B} = [\mathbf{A}^{(1)} \otimes \mathbf{B}^{(1)}, \cdots, \mathbf{A}^{(Q)} \otimes \mathbf{B}^{(Q)}]$.

$$\mathbf{X}_3 = (\mathbf{A}\boldsymbol{\Psi} \diamond \mathbf{B}\boldsymbol{\Phi})\mathbf{C}^T = (\mathbf{A} \,|\otimes|\, \mathbf{B})F(\boldsymbol{\Psi}, \boldsymbol{\Phi})\mathbf{C}^T.$$

$$F(\boldsymbol{\Psi}, \boldsymbol{\Phi}) = BlockDiag(\underbrace{\boldsymbol{\Psi}^{(1)} \diamond \boldsymbol{\Phi}^{(1)}}_{\mathbf{I}_{R_3^{(1)}}} \cdots \underbrace{\boldsymbol{\Psi}^{(Q)} \diamond \boldsymbol{\Phi}^{(Q)}}_{\mathbf{I}_{R_3^{(Q)}}})$$

$$\Rightarrow \mathbf{X}_3 = (\mathbf{A} \,|\otimes|\, \mathbf{B})\mathbf{G}_3\mathbf{C}^T, \quad \mathbf{G}_3 = \mathbf{I}_{R_3}$$

(*Constrained Block-Tucker3 decomp.*)

Constrained Block-PARAFAC *linked to* Block-Tucker3

- Unfolded matrices $\mathbf{X}_1$ and $\mathbf{X}_2$:

$$\mathbf{X}_1 = (\mathbf{C} \,|\otimes|\, \mathbf{A})\mathbf{G}_1\mathbf{B}^T, \qquad \mathbf{X}_2 = (\mathbf{B} \,|\otimes|\, \mathbf{C})\mathbf{G}_2\mathbf{A}^T,$$

- Unfolded block-cores $\mathbf{G}_1$ and $\mathbf{G}_2$:

$$\mathbf{G}_1 = BlockDiag(\mathbf{G}_1^{(1)} \cdots \mathbf{G}_1^{(Q)}) \in \mathbb{C}^{R' \times R_2},$$

$$\mathbf{G}_2 = BlockDiag(\mathbf{G}_2^{(1)} \cdots \mathbf{G}_2^{(Q)}) \in \mathbb{C}^{R'' \times R_1},$$

$$R' = \sum_{q=1}^{Q} R_1^{(q)} R_3^{(q)}, \quad R'' = \sum_{q=1}^{Q} R_2^{(q)} R_3^{(q)}$$

$$\mathbf{G}_1^{(q)} = (\mathbf{I}_{R_3^{(q)}} \diamond \mathbf{\Psi}^{(q)})\mathbf{\Phi}^{(q)T}, \quad \mathbf{G}_2^{(q)} = (\mathbf{\Phi}^{(q)} \diamond \mathbf{I}_{R_3^{(q)}})\mathbf{\Psi}^{(q)T}$$

- Are $\mathbf{G}_1^{(q)}$ and $\mathbf{G}_2^{(q)}$ column-wise orthogonal ?

Constrained Block-PARAFAC *linked to* Block-Tucker3

- We have:

$$\mathbf{G}_1^{(q)T}\mathbf{G}_1^{(q)} = \mathbf{\Phi}^{(q)}\mathbf{\Phi}^{(q)T} = R_1^{(q)}\mathbf{I}_{R_2^{(q)}}$$

$$\mathbf{G}_2^{(q)T}\mathbf{G}_2^{(q)} = \mathbf{\Psi}^{(q)}\mathbf{\Psi}^{(q)T} = R_2^{(q)}\mathbf{I}_{R_1^{(q)}}$$

- Column-wise orthogonality also for $\mathbf{G}_1$ and $\mathbf{G}_2$

Constraint matrices from a PARAFAC perspective give rise to orthogonal tensor cores from a Tucker3 perspective

Uniqueness issues

- Uniqueness proof of constrained Block-PARAFAC relies on Harshman's original proof of "minimum conditions" for PARAFAC [Harshman'72].
- The proof sheds light on the between-block resolution/separability for constrained Block-PARAFAC.
- Within-block uniqueness can be studied separately for each block, possibly taking special (within-block) structures into account.
- Different levels of "partial" uniqueness are possible for constrained Block-PARAFAC

Uniqueness issues

Theorem: Assuming full rank $\mathbf{A}^{(q)} \in \mathbb{C}^{I_1 \times R_1^{(q)}}$, $\mathbf{B}^{(q)} \in \mathbb{C}^{I_2 \times R_2^{(q)}}$ and $\mathbf{C}^{(q)} \in \mathbb{C}^{I_3 \times R_1^{(q)} R_2^{(q)}}$, $q = 1, \ldots, Q$, and linear independency of every set $\{\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(Q)}\}$, $\{\mathbf{B}^{(1)}, \ldots, \mathbf{B}^{(Q)}\}$ and $\{\mathbf{C}^{(1)}, \ldots, \mathbf{C}^{(Q)}\}$, if:

$$I_1 I_2 \geq \sum_{q=1}^{Q} R_1^{(q)} R_2^{(q)}, \quad I_1 I_3 \geq \sum_{q=1}^{Q} R_2^{(q)}, \quad I_2 I_3 \geq \sum_{q=1}^{Q} R_1^{(q)}.$$

between-block uniqueness is achieved and $\overline{\mathbf{A}} = \mathbf{A}\mathbf{T}_a$, $\overline{\mathbf{B}} = \mathbf{B}\mathbf{T}_b$ and $\overline{\mathbf{C}} = \mathbf{C}\mathbf{T}_c$ (up to permutation ambiguities).

$$\begin{aligned} \mathbf{T}_a &= BlockDiag(\mathbf{T}_a^{(1)} \cdots \mathbf{T}_a^{(Q)}), \\ \mathbf{T}_b &= BlockDiag(\mathbf{T}_b^{(1)} \cdots \mathbf{T}_b^{(Q)}), \\ \mathbf{T}_c &= BlockDiag(\mathbf{T}_c^{(1)} \cdots \mathbf{T}_c^{(Q)}), \end{aligned}$$

$$\left(\mathbf{T}_a^{(q)} \otimes \mathbf{T}_b^{(q)}\right)^{-1} = \mathbf{T}_c^{(q)T}, \quad q = 1, \ldots, Q$$

Uniqueness issues

- Rotational freedom confined within the blocks;
  (due to block-diagonal structure of $\mathbf{T}_a$, $\mathbf{T}_b$ and $\mathbf{T}_c$)

- Within-block rotational freedom is constrained;
  (not as complete as for Tucker3)

- Recovering of complete within-block uniqueness (q-th block):
  * If $\mathbf{C}^{(q)}$ is known $\Rightarrow \mathbf{T}_a^{(q)} \otimes \mathbf{T}_b^{(q)} = \mathbf{I}$;
  * If rotational indeterminacy $\mathbf{T}_c^{(q)}$ is fixed;

- Partial uniqueness arises if a subset of $\{\mathbf{T}_a^{(1)}, \ldots, \mathbf{T}_a^{(Q)}\}$ or $\{\mathbf{T}_b^{(1)}, \ldots, \mathbf{T}_b^{(Q)}\}$ is fixed.

Uniqueness issues

Blocks with equal interaction patterns

$R_1^{(1)} = \ldots = R_1^{(Q)} = \overline{R}_1$ and $R_2^{(1)} = \ldots = R_2^{(Q)} = \overline{R}_2$

- Equivalent necessary condition:

$$min\left(\lfloor\frac{I_1 I_2}{\overline{R}_1\overline{R}_2}\rfloor, \lfloor\frac{I_1 I_3}{\overline{R}_2}\rfloor, \lfloor\frac{I_2 I_3}{\overline{R}_1}\rfloor\right) \geq Q.$$

- For $\overline{R}_1 = \overline{R}_2 = 1$ (standard PARAFAC) the above condition reduces to the necessary uniqueness conditions of [Liu-Sid'01].

Partial uniqueness in constrained Block-PARAFAC

- Some blocks can be uniquely determined, while the remaining ones are either nonunique or partially unique
  $\Rightarrow$ (breaking down of uniqueness **between** blocks)

- For the remaining (no strictly unique) blocks:

  Situation 1: Within-block nonuniqueness
  All the corresponding component matrices affected by unknown rotational indeterminacy;

  Situation 2: Within-block partial uniqueness
  Some component matrices (or a subset of their columns) uniquely determined $\Rightarrow$ (breaking down of uniqueness **within** a block)

Partial uniqueness in constrained Block-PARAFAC

Example 1:

$Q = 3$ blocks

$$\{R_1^{(1)}, R_2^{(1)}\} = \{1,1\}, \quad \{\mathbf{a}^{(1)}, \mathbf{b}^{(1)}, \mathbf{c}^{(1)}\}$$
$$\{R_1^{(2)}, R_2^{(2)}\} = \{2,2\}, \quad \{\mathbf{A}^{(2)}, \mathbf{B}^{(2)}, \mathbf{C}^{(2)}\}$$
$$\{R_1^{(3)}, R_2^{(3)}\} = \{1,2\}, \quad \{\mathbf{a}^{(3)}, \mathbf{B}^{(3)}, \mathbf{C}^{(3)}\}$$

Block-constraint matrices:

$$\mathbf{\Psi} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{\Phi} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Block 1: *unique* / Block 2: *nonunique* / Block 3: *partially unique*

$$/ \left(\mathbf{T}_a^{(2)} \otimes \mathbf{T}_b^{(2)}\right)^{-1} = \mathbf{T}_c^{(2)T} / \qquad \mathbf{T}_b^{(3)T} = \mathbf{T}_c^{(3)-1}$$

Partial uniqueness in constrained Block-PARAFAC

Example 2:

$Q = 3$ blocks

$$\{R_1^{(1)}, R_2^{(1)}\} = \{1, 2\}, \quad \{\mathbf{a}^{(1)}, \mathbf{B}^{(1)}, \mathbf{C}^{(1)}\}$$
$$\{R_1^{(2)}, R_2^{(2)}\} = \{1, 2\}, \quad \{\mathbf{a}^{(2)}, \mathbf{B}^{(2)}, \mathbf{C}^{(2)}\}$$
$$\{R_1^{(3)}, R_2^{(3)}\} = \{1, 2\}, \quad \{\mathbf{a}^{(3)}, \mathbf{B}^{(3)}, \mathbf{C}^{(3)}\}$$

Block-constraint matrices:

$$\boldsymbol{\Psi} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{\Phi} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ *Partial uniqueness in all the blocks*

$\Rightarrow$ *Complete uniqueness in the first-mode*

This case coincides with the “FIA PARALIND model" [Bro-Harsh-Sid’05]

Applications in Wireless Signal Processing

- Some previous works using PARAFAC modeling in wireless communications:
  (*DS-CDMA, OFDM, blind beamforming, multiantenna (MIMO) systems*,...)

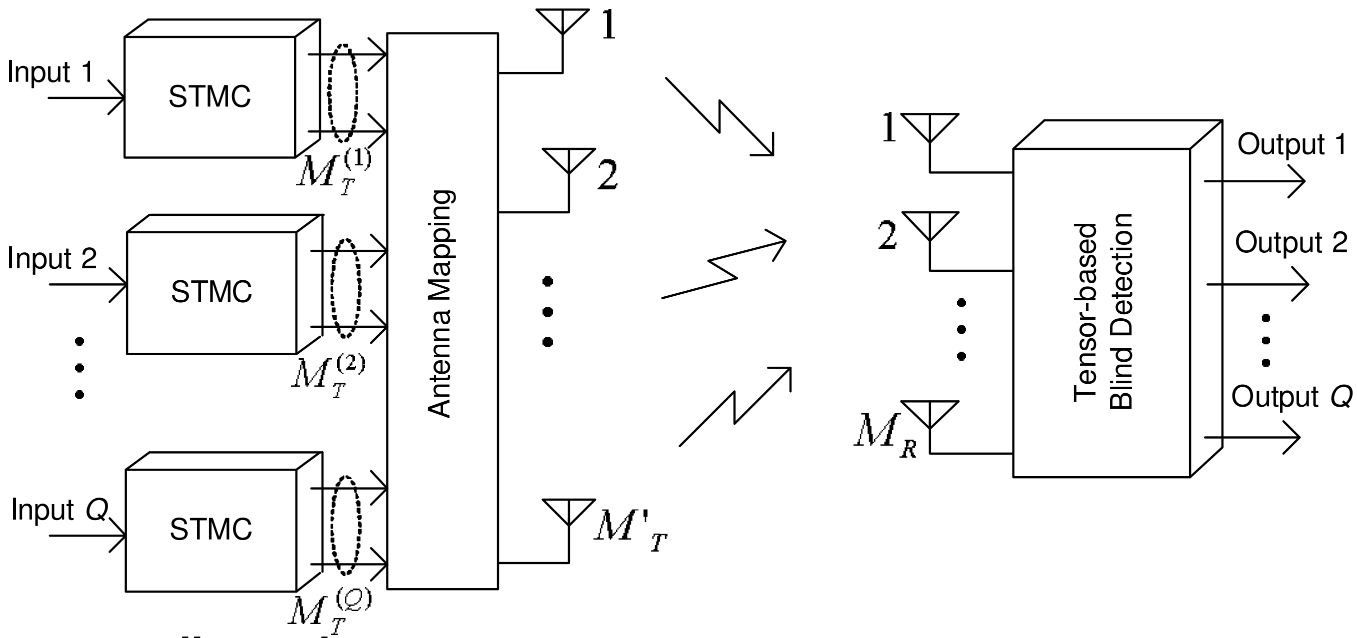
  [Sidiropoulos et al.'00-1] [Sidiropoulos et al.'00-2] [Sidiropoulos&Dimic'01], [Sidiropoulos&Liu'01] [Sidiropoulos&Budampati'02] [Jiang&Sidiropoulos'03] [de Baynast&De Lathauwer'03] [de Baynast et al.'03] [De Lathauwer'05]

- Two classes of wireless communication problems formulated using constrained Block-PARAFAC modeling:
  - ∗ Multiantenna coding
    (with spatial spreading and single-antenna multiplexing)
  - ∗ Blind beamforming
    (under specular propagation and large delay spread)

- Constrained Block-PARAFAC structure arises in some of previous works while generalizing some previously proposed three-way models

Application 1: Multiantenna coding

Space-Time Multiplexing Coding (STMC)



Three-way array dimensions:

$I_1 = M_R$: Nb. of receive antennas

$I_2 = N$: Nb. of time-slots

$I_3 = P$: Nb. of coded symbols per time-slot (code length)

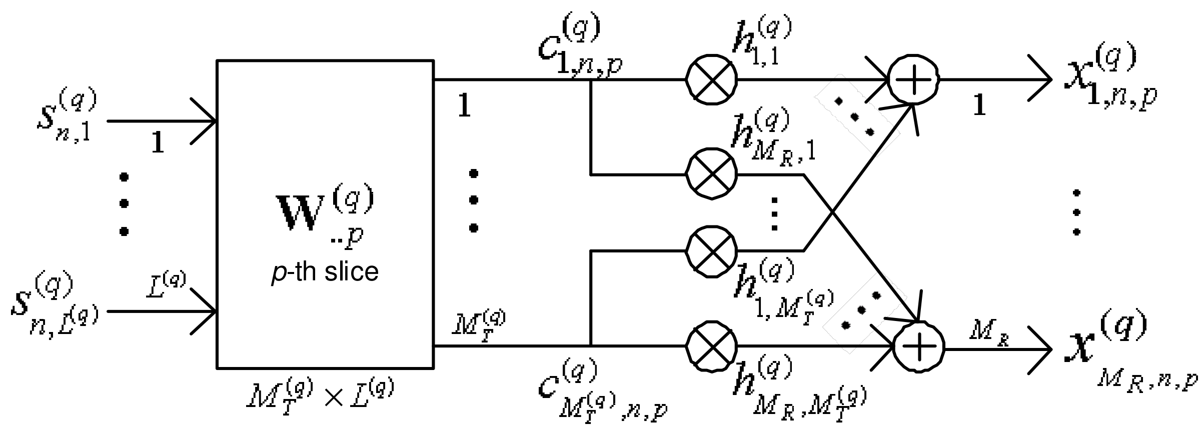
Model parameters:

Q: Nb. of transmission groups (or users to be served)

$R_1^{(q)} = L^{(q)}$: Nb. spatially-multiplexed signals

$R_2^{(q)} = M_T^{(q)}$: Nb. of spreading transmit antennas

Application 1: Multiantenna coding



- Decomposition of the received signal as a three-way array:

$$x_{m_R,n,p} = \sum_{q=1}^{Q} \sum_{m_T^{(q)}=1}^{M_T^{(q)}} h_{m_R,m_T^{(q)}}^{(q)} \sum_{l^{(q)}=1}^{L^{(q)}} s_{n,l^{(q)}}^{(q)} w_{m_T^{(q)},l^{(q)},p}^{(q)} + v_{m_R,n,p}$$

$x_{m_R,n,p} = [\mathcal{X}]_{m_R,n,p}$: *received signal*

$h_{m_R,m_T^{(q)}}^{(q)} = [\mathbf{H}^{(q)}]_{m_R,m_T^{(q)}}$: *q-th MIMO channel*

$s_{n,l^{(q)}}^{(q)} = [\mathbf{S}^{(q)}]_{n,l^{(q)}}$: *transmitted symbols (during the n-th time-slot)*

$w_{m_T^{(q)},l^{(q)},p}^{(q)} = [\mathbf{W}^{(q)}]_{p,(l^{(q)}-1)M_T^{(q)}+l^{(q)}}$: *coding/multiplexing tensor*

Application 1: Multiantenna coding

* $L' = L^{(1)} + \cdots + L^{(Q)}$: *total number of multiplexed signals*
* $M'_T = M_T^{(1)} + \cdots + M_T^{(Q)}$: *total number of transmit antennas*
* $R' = L^{(1)} M_T^{(1)} + \cdots + L^{(Q)} M_T^{(Q)}$: *number of columns of* $\mathbf{W}$

- Constrained Block-PARAFAC model:
  $(\mathbf{H} \in \mathbb{C}^{M_R \times M'_T}, \quad \mathbf{S} \in \mathbb{C}^{N \times L'}, \quad \mathbf{W} \in \mathbb{C}^{P \times R'})$

$$\mathbf{X} = (\mathbf{H\Psi} \diamond \mathbf{S\Phi})\mathbf{W}^T + \mathbf{V}, \qquad \mathbf{WW}^H = M'_T \mathbf{I}_P$$

- Constrained Block-PARAFAC model covers some multi-antenna coding schemes as special cases:

  * $L^{(q)} = M_T^{(q)} = 1, \quad Q > 1$:

    Reduces to the multiantenna code of [Sidiropoulos&Budampati'02]

  * $Q = 1, \quad L^{(q)} > 1, M_T^{(q)} > 1$:

    Takes the form of the multiantenna code of [Hassibi'02]

Constraint matrices: *Physical interpretation*

- $\mathbf{\Psi}$ and $\mathbf{\Phi}$ can be seen as *symbol-to-antenna loading matrices*
- Reveal the *joint spreading-multiplexing pattern* considered at the transmitter, for each transmission group;
- By configuring the joint pattern of 1's and 0's of these matrices $\longrightarrow$ different multiantenna coding schemes can be constructed

Example:

$\overline{M}'_T = 3$ transmit antennas, $Q = 2$ Tx groups.

Spreading-multiplexing structures: $(M_T^{(1)}, L^{(1)}) = (2, 1)$ and $(M_T^{(2)}, L^{(2)}) = (1, 3)$.

$$\mathbf{\Psi} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \in \mathbb{C}^{M'_T \times R'}, \quad \mathbf{\Phi} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{C}^{L' \times R'}$$

1) Rows of $\mathbf{\Psi}$ reveal the spatial *multiplexing factor*

$\Rightarrow$ nb. of symbols simultaneously loaded at the same transmit antenna

2) Rows of $\mathbf{\Phi}$ reveal the spatial *spreading factor*

$\Rightarrow$ nb. of transmit antennas simultaneously transmitting the same symbol

Constraint matrices: *Physical interpretation*

Joint spreading-multiplexing pattern: The matrix product $\boldsymbol{\Psi}\boldsymbol{\Phi}^T$

$$\boldsymbol{\Psi}\boldsymbol{\Phi}^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \in \mathbb{C}^{M'_T \times L'}$$

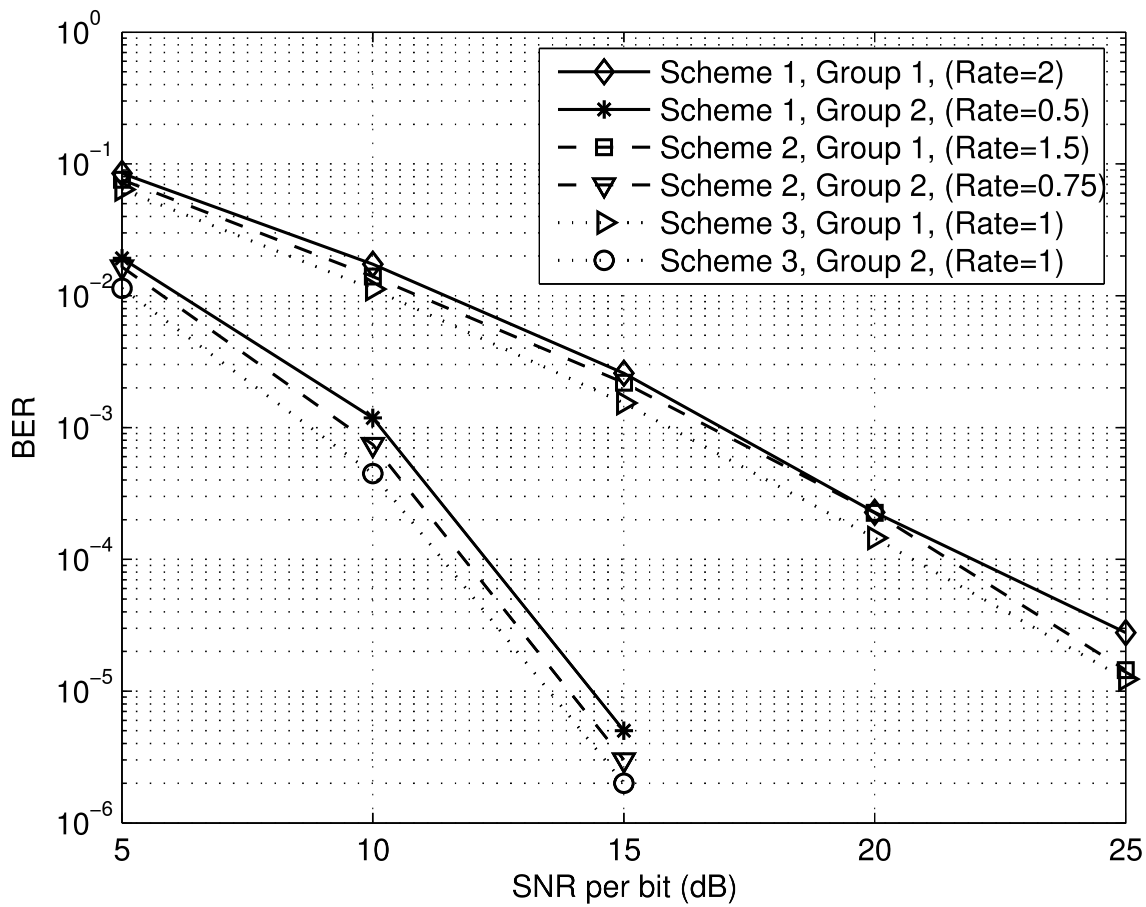
- Reading $\boldsymbol{\Psi}\boldsymbol{\Phi}^T$ column-wise (for a fixed row):
  - $*$ check for the nb. of data-streams multiplexed at a given antenna
- Reading $\boldsymbol{\Psi}\boldsymbol{\Phi}^T$ row-wise (for a fixed column):
  - $*$ check for the nb. of antennas spreading a given data-stream

Remark: Symbol-to-antenna allocation obtained by permuting the columns of the constraint matrices $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$. Important in spatially correlated channels

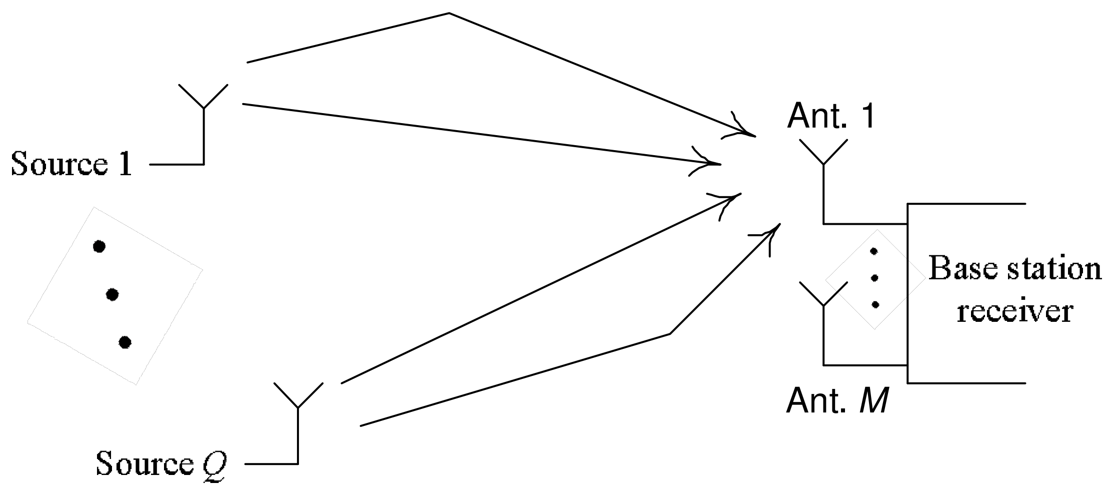
Constrained Block-PARAFAC based multi-antenna receiver

	Scheme 1	Scheme 2	Scheme 3
$(M^{(1)}, L^{(1)})$	(1,4)	(1,2)	(1,1)
$(M^{(2)}, L^{(2)})$	(2,1)	(2,1)	(2,1)

Rate (q-th group) = $(L^{(q)}/P)log_2(\mu)$ bits/channel use



Application 2: Blind Beamforming



Three-way array dimensions:

$I_1 = M$: Nb. of receiver antennas
$I_2 = N$: Nb. of symbols periods
$I_3 = P$: Oversampling factor (nb. of samples/symbol period)

Model parameters:

Q: Nb. of source signals
$R_1^{(q)} = L^{(q)}$: Nb. of multipaths (q-th source); $L' = L^{(1)} + \cdots + L^{(Q)}$
$R_2^{(q)} = K$: Temporal support of the convolutive channel (common for all sources)

Application 2: Blind Beamforming

- Decomposition of the received signal as a three-way array:

$$x_{m,n,p} = \sum_{q=1}^{Q} \sum_{l^{(q)}=1}^{L^{(q)}} b_{l^{(q)}}^{(q)} a_{m,l^{(q)}}^{(q)} \sum_{k=1}^{K} s_{n,k}^{(q)} g_{l^{(q)},k,p}^{(q)} + v_{m,n,p}$$

$x_{m,n,p} = [\mathcal{X}]_{m,n,p}$: *received signal*

$b_{l^{(q)}}^{(q)} = [Diag(\mathbf{B}^{(q)})]_{l^{(q)},l^{(q)}}$: *multipath gains/amplitudes*

$a_{m,l^{(q)}}^{(q)} = a_m^{(q)}(\theta_{l^{(q)}}) = [\mathbf{A}^{(q)}]_{m,l^{(q)}}$ *array response (Vandermonde structure)*

$g_{l^{(q)},k,p}^{(q)} = g(k-1+(p-1)/P - \tau_{l^{(q)}}) = [\mathbf{G}^{(q)}]_{p,(l^{(q)}-1)K+k}$ *pulse shape*

$s_{n,k}^{(q)} = [\mathbf{S}^{(q)}]_{n,k}$: *transmitted symbols (Toeplitz structure)*

- Constrained Block-PARAFAC model:
  $(\mathbf{A} \in \mathbb{C}^{M\times L'}, \quad \mathbf{S} \in \mathbb{C}^{N\times QK}, \quad \mathbf{H} \in \mathbb{C}^{P\times L'K})$

$$\mathbf{X} = (\mathbf{A}\mathbf{\Psi} \diamond \mathbf{S}\mathbf{\Phi})\mathbf{H}^T + \mathbf{V}, \quad \mathbf{H} = \mathbf{G}(\mathbf{B} \otimes \mathbf{I}_K)$$

$$\mathbf{\Psi} = BlockDiag(\mathbf{I}_{L^{(1)}} \otimes \mathbf{1}_K^T \cdots \mathbf{I}_{L^{(Q)}} \otimes \mathbf{1}_K^T)$$

$$\mathbf{\Phi} = BlockDiag(\mathbf{1}_{L^{(1)}}^T \otimes \mathbf{I}_K \cdots \mathbf{1}_{L^{(Q)}}^T \otimes \mathbf{I}_K)$$

Special cases of constrained Block-PARAFAC

Special case 1: *Far-field reflections*

$(a_{m,1}^{(q)} \approx \cdots \approx a_{m,L^{(q)}}^{(q)} = a_m^{(q)}, \quad q = 1, \ldots, Q)$

$$x_{m,n,p} = \sum_{q=1}^{Q} a_m^{(q)} \sum_{k=1}^{K} h'^{(q)}_{p,k} s^{(q)}_{n,k} + v_{m,n,p}, \quad \text{with} \quad h'^{(q)}_{p,k} = \sum_{l^{(q)}=1}^{L^{(q)}} b^{(q)}_{l^{(q)}} g^{(q)}_{l^{(q)},k,p}$$

- Constrained Block-PARAFAC model:
  $(\mathbf{A}' \in \mathbb{C}^{M \times Q}, \quad \mathbf{S} \in \mathbb{C}^{N \times QK}, \quad \mathbf{H}' \in \mathbb{C}^{P \times QK})$

$$\mathbf{X} = (\mathbf{A}'\boldsymbol{\Psi} \diamond \mathbf{S})\mathbf{H}'^T + \mathbf{V}, \qquad \boldsymbol{\Psi} = \mathbf{I}_Q \otimes \mathbf{1}_K^T, \quad \boldsymbol{\Phi} = \mathbf{I}_{QK}$$

$$\text{with} \quad \mathbf{A}' = [\mathbf{a}^{(1)} \cdots \mathbf{a}^{(Q)}], \quad \mathbf{H}' = \mathbf{G}\mathbf{B}\mathbf{J}_g,$$

$$\mathbf{J}_g = BlockDiag(\mathbf{1}_{L^{(1)}} \otimes \mathbf{I}_K \cdots \mathbf{1}_{L^{(Q)}} \otimes \mathbf{I}_K)$$

Remark: This model arises in [Sidiropoulos&Dimic'01] and [de Baynast&De Lathauwer'03] under different formulations and terminologies

∗ In [Sidiropoulos&Dimic'01]: PARALIND model

∗ In [de Baynast&De Lathauwer'03]: Generalized CP model

Special cases of constrained Block-PARAFAC

Special case 2: *Local scattering (small delay spread)*

$(max(\tau_{lq}) << T,\, q = 1, \ldots, Q,\;\; K = 1)$

$$x_{m,n,p} = \sum_{q=1}^{Q} \sum_{l^{(q)}=1}^{L^{(q)}} b_{l^{(q)}}^{(q)} a_{m,l^{(q)}}^{(q)} g_{p,l^{(q)}}^{(q)} s_n^{(q)} + v_{m,n,p}$$

- Constrained Block-PARAFAC model:
  $(\mathbf{A} \in \mathbb{C}^{M \times L'}, \quad \mathbf{S} \in \mathbb{C}^{N \times Q}, \quad \mathbf{H}'' \in \mathbb{C}^{P \times L'})$

$$\mathbf{X} = (\mathbf{A} \diamond \mathbf{S}\mathbf{\Phi})\mathbf{H}''^{T} + \mathbf{V}, \quad \mathbf{\Psi} = \mathbf{I}_{L'}, \quad \mathbf{\Phi} = BlockDiag(\mathbf{1}_{L^{(1)}}^{T} \cdots \mathbf{1}_{L^{(Q)}}^{T})$$

with $\mathbf{G} = [\mathbf{g}_1^{(1)} \cdots \mathbf{g}_{l^{(q)}}^{(q)} \cdots \mathbf{g}_{L^{(Q)}}^{(Q)}] \in \mathbb{C}^{P \times L'}, \quad \mathbf{H}'' = \mathbf{G}\mathbf{B},$

$$\mathbf{J}_g = BlockDiag(\mathbf{1}_{L^{(1)}} \otimes \mathbf{I}_K \cdots \mathbf{1}_{L^{(Q)}} \otimes \mathbf{I}_K)$$

Remark: This special case was considered in [Sidiropoulos&Liu'01].

Constrained Block-PARAFAC based receiver

Goal:

1) Separate the Q source contributions (ensure between-block uniqueness)

2) Recover each source signal (ensure within-block partial uniqueness)

ALS + Subspace+ FA algorithm:

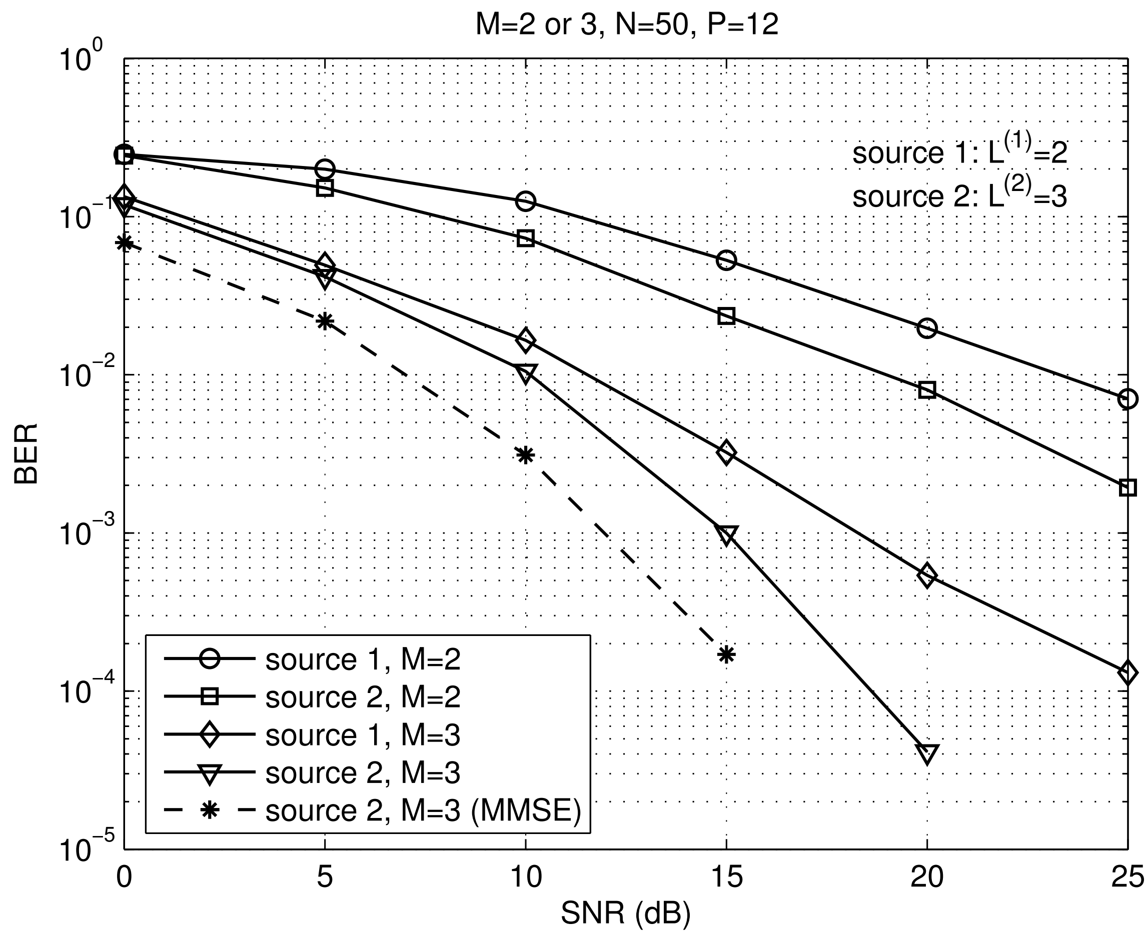
- Iterative combination of Alternating Least Squares (ALS), Subspace method, and Finite Alphabet (FA) projection:
  - $*$ ALS + FA steps:

    Separate the Q source signals;
  - $*$ Subspace step:

    Recover the transmitted sequences by fixing a rotational ambiguity matrix.
- Same idea of [Sidiropoulos&Dimic'01] and [de Baynast&De Lathauwer'03], but fitting a different three-way model.
- Forcing the FA property on the symbol matrix accelerates convergence (although not optimal)

Constrained Block-PARAFAC based receiver

	angles-of-arrival	time-delays
Source 1	$(\theta_1^{(1)}, \theta_2^{(1)}) = (-50°, -20°)$	$(\tau_1^{(1)}, \tau_2^{(1)}) = (0, T)$
Source 2	$(\theta_1^{(2)}, \theta_2^{(2)}, \theta_3^{(2)}) = (0°, 30°, 50°)$	$(\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)}) = (0, 0.2T, T)$



Concluding remarks and perspectives

- Constrained Block-PARAFAC decomposition: Based on De Lathauwer's block-tensor approach, but formulated using constraint matrices
- Enjoys between-block uniqueness/resolution and different levels of within-block partial uniqueness
- Application of constrained Block-PARAFAC to two wireless communication problems: multiantenna coding and blind beamforming
- Constraint matrices are meaningful in wireless signal processing applications (e.g. multiantenna coding)

Perspectives:

- Within-block uniqueness from a constrained Block-Tucker3 point of view;
- Applications aspects:

  (i) Robustness to under- and over-parameterizations (blind beamforming);

  (ii) Optimization of the constraint matrices (multiantenna coding)